Noncoherent Integration Gain, and its Approximation

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1 Signal-to-Noise Ratio and Integration in Radar Signal Processing

Signal-to-noise ratio (SNR) is a fundamental determinant of the quality of many radar signal processing operations. To cite some important examples [1]:

- for a given probability of false alarm P_{FA}, the probability of detection, P_D, increases as SNR increases
- measurement precision in range, angle, and Doppler frequency improves (standard deviation of measurement error decreases) as SNR increases
- image contrast (dynamic range) in synthetic aperture radar (SAR) imaging increases as SNR increases.

Many radar signal processing operations seek to increase SNR and thus radar performance by adding ("integrating") multiple data samples together. Two major classes of integration are recognized: coherent and noncoherent.

2 Coherent Integration Gain

Coherent integration gain is the increase in SNR obtained by coherently integrating multiple measurements of a signal in additive noise [1]. The integration is considered "coherent" when it is performed on the complex data, so that both the amplitude and the phase of the data are utilized. Consider complex data $x[n]$ comprised of a complex constant signal $s[n] = Ae^{j\phi}$ (independent of the index n) and additive zero complex white Gaussian noise $w[n]$ of variance σ_w^2:

$$x[n] = s[n] + w[n] = Ae^{j\phi} + w[n] \tag{1}$$

The SNR of a single sample of x, denoted χ_1, is

$$\chi_1 = \frac{A^2}{\sigma_w^2} \tag{2}$$

Consider the sum of N such samples:

$$z \equiv \sum_{n=0}^{N-1} x[n] = \sum_{n=0}^{N-1} \left(s[n] + w[n]\right) = \sum_{n=0}^{N-1} \left(Ae^{j\phi} + w[n]\right) = NAe^{j\phi} + \sum_{n=0}^{N-1} w[n] \tag{3}$$

The sum z is still in the form of the sum of a signal component $NAe^{j\phi}$ and a noise component that is the sum of N noise samples. The power in the signal component is clearly $(NA)^2$. The power in the noise component is

$$\begin{aligned}\mathbf{E}\left\{\left|\sum_{n=0}^{N-1} w[n]\right|^2\right\} &= \mathbf{E}\left\{\left(\sum_{n=0}^{N-1} w[n]\right)\left(\sum_{l=0}^{N-1} w^*[l]\right)\right\} \\ &= \sum_{n=0}^{N-1}\sum_{l=0}^{N-1} \mathbf{E}\left\{w[n]w^*[l]\right\} \\ &= N\sigma_w^2\end{aligned} \tag{4}$$

where we have relied on $w[n]$ being a white random process. The SNR of z, χ_{N_c}, is

$$\chi_{N_c} = \frac{(NA)^2}{N\sigma_w^2} = \frac{NA^2}{\sigma_w^2} = N\chi_1 \tag{5}$$

This increase in SNR by a factor of N due to coherent integration of N data samples is called the *coherent integration gain,* and is denoted here as G_c:

$$G_c = \frac{\chi_1}{\chi_{N_c}} \tag{6}$$

Obtaining this gain obviously depended on having the signal samples add in phase (sometimes called adding on a "voltage basis") so that the signal component power increased by a factor of N^2. In contrast, the power of the integrated noise increased only by a factor of N (a result sometimes called adding on a "power basis"). It should also be clear that the signal component can be generalized to be of the form $A\exp(j\phi[n])$, thus allowing for a phase modulation, provided that the phase modulation is compensated during integration so that the signal components again add in phase:

$$\begin{aligned}z_c &\equiv \sum_{n=0}^{N-1}\left(s[n]+w[n]\right)e^{-j\phi[n]} = \sum_{n=0}^{N-1}\left(Ae^{j\phi[n]}+w[n]\right)e^{-j\phi[n]} \\ &= NA + \sum_{n=0}^{N-1} w[n]e^{-j\phi[n]}\end{aligned} \tag{7}$$

Clearly, the phase sequence $\phi[n]$ must be known or estimated for compensation to be applied successfully. This generalization of coherent integration is the common thread underlying the processing gain achieved by any matched filtering technique, such as Doppler filtering, synthetic aperture radar (SAR) image formation, and space-time adaptive processing (STAP).

To compute the effect of coherent integration on detection performance, results for a single sample of signal in noise are used, with χ_{N_c} replacing χ_1 [1][2]. For example, if an SNR of 15 dB is required to achieve a specified P_D and P_{FA} with a single data sample, the same P_D and P_{FA} can be achieved by collecting and coherently integrating 2 samples having individual SNRs of 12 dB (a reduction by a factor of 2, equivalent to 3 dB), or 10 samples with individual SNRs of 5 dB (a reduction by a factor of 10, equivalent to 10 dB).

3 Noncoherent Integration Gain

In noncoherent integration, the summation is applied to a function of the magnitude of the complex data sample $x[n]$, thus discarding the phase information before integration. Integration is typically performed on $|x[n]|$, $|x[n]|^2$, or $\log(|x[n]|)$. These are commonly referred to respectively as linear, square-law, or log detected data. Assuming for example a linear detector, noncoherent integration consists of forming the sum

$$z_{nc} \equiv \sum_{n=0}^{N-1} \left| x[n] \right| = \sum_{n=0}^{N-1} \left| s[n] + w[n] \right| = \left| \left(Ae^{j\phi} + w[n] \right) \right| \tag{8}$$

However, it is not possible to define a signal-to-noise ratio in the noncoherent case because z_{nc} cannot be decomposed into the sum of distinct signal-only and noise-only components as was done in Eq. (3). This is because the nonlinear transformations $|\cdot|$, $|\cdot|^2$, and $\log|\cdot|$ create cross-products of signal and noise components. Thus, it is not possible to directly compute an SNR after noncoherent integration, and in turn it is not possible to directly compute a noncoherent integration gain.

However, a noncoherent integration gain G_{nc} can be defined indirectly by considering the single-sample SNR required to achieve a specified performance in some signal processing problem when noncoherently integrating multiple data samples, and comparing that to the SNR required to achieve the same performance when only a single sample is used. Denote the single-sample SNR needed to achieve a specified performance when N such samples are noncoherently integrated as $\chi_{1,N}$. Now consider the detection curves in Figure 1, adapted from [1]. These show the P_D achieved as a function of SNR with a linear detector and noncoherent integration for a P_{FA} of 10^{-8}. Such curves define an implicit noncoherent integration gain. For instance, consider the single sample SNR required to achieve $P_D = 0.8$ for $N = 1$ and again for $N = 10$. From the figure, it can be seen that the single-sample SNR required for $N = 1$ is $\chi_1 = 14$ dB, while for $N = 10$ the SNR required is reduced to $\chi_{1,10} = 5.7$ dB. The difference of 8.3 dB, a factor of about 6.8, is the noncoherent integration gain G_{nc}. In general,

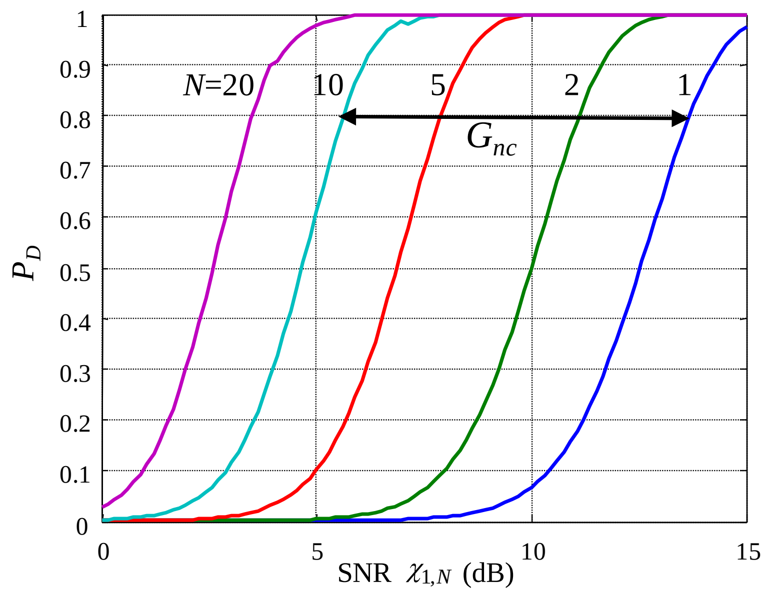

Figure 1. Illustration of effect of noncoherent integration on single-sample SNR required. $P_{FA} = 10^{-8}$.

$$G_{nc} = \frac{\chi_1}{\chi_{1,N}} \tag{9}$$

It is often stated that $G_{nc} \approx \sqrt{N} = N^{0.5}$. This example shows that is not the case. Specifically, for this example $G_{nc} = 6.8 = 10^{0.833}$. (Note that the coherent integration gain $G_c = N^1$). When expressed in the form N^{α}, it is the case that $0.5 < \alpha < 1$ for noncoherent integration gain. Consequently, noncoherent integration is not as *efficient* as coherent integration in the sense that it takes a larger value of N to achieve a given integration gain than is required for coherent integration. We will return to integration efficiency in Section 7.

4 Computing Noncoherent Integration Gain for Detection

Because the detectors used to obtain noncoherent data are nonlinear operations, G_{nc} is not easily calculated directly. The most straightforward way to compute G_{nc} in the context of detection is to use results for calculation of detection performance for a given problem to calculate the single-sample SNR χ_1 required when using $N = 1$ sample; repeat for $N > 1$ samples to get $\chi_{1,N}$; and then compute the ratio $\chi_1/\chi_{1,N}$ of Eq. (9).

Consider the common radar case of a nonfluctuating target in complex Gaussian noise, sometimes called the *Marcum* or *Swerling 0* model. For a given P_{FA}, single-sample

SNR $\chi_{1,N}$, number of samples noncoherently integrated N, and a square-law detector, P_D is found with the pair of equations [1][2].

$$P_{FA} = 1 - I[\frac{T}{\sqrt{N}}, N-1] \tag{10}$$

and

$$P_D = Q_M(\sqrt{2N\chi_{1,N}}, \sqrt{2T}) + \underbrace{e^{-\left(T+N\chi_{1,N}\right)} \sum_{r=2}^{N} \left(\frac{T}{N\chi_{1,N}}\right)^{\frac{r-1}{2}} I_{r-1}\left(2\sqrt{NT\chi_{1,N}}\right)}_{\text{(only contributes when } N > 2)} \tag{11}$$

where $I[\cdot,\cdot]$ is Pearson's form of the incomplete gamma function and $Q_M(\cdot,\cdot)$ is Marcum's Q function [1]. Equation (10) is solved for the threshold T required to achieve the desired P_{FA}; T is then used with N to solve Eq. (11) for the SNR $\chi_{1,N}$ that gives the desired P_D. The same process can be used for the other Swerling models by replacing Eq. (11) with the appropriate equation for each fluctuation model. (Equation (10) is unchanged.) These solutions must generally be obtained numerically. One set of MATLAB software for doing so is available at [3].

5 Approximations to Noncoherent Integration Gain for Detection of Nonfluctuating Targets

Albersheim's equation is an empirical approximation to the nonfluctuating target detection problem, but for a linear detector [4][5]. The single-sample SNR is computed according to

$$\begin{aligned} &A = \ln\left(\frac{0.62}{P_{FA}}\right), \quad B = \ln\left(\frac{P_D}{1-P_D}\right) \\ &\chi_{1,N} = -5\log_{10} N + \left(6.2 + \left(\frac{4.54}{\sqrt{N+0.44}}\right)\right)\log_{10}\left(A + 0.12AB + 1.7B\right) \quad \text{dB} \end{aligned} \tag{12}$$

Noncoherent integration gain is easily obtained by computing the single-sample SNR using Albersheim's equation with different values of N. Because the SNR required to achieve a given detection performance with a linear vs. a square-law detector varies by only about 0.2 dB over a wide range of parameters [6], Albersheim's equation is generally useful for both detector types.

Peebles has presented the following empirical formula that gives noncoherent integration gain directly for the nonfluctuating target case and a square-law detector [6]:

$$G_{nc}\ \text{(dB)} = 6.79\left(1+0.253P_D\right)\left(1+\frac{\log_{10}\left(1/P_{FA}\right)}{46.6}\right)\left(\log_{10} N\right)\cdot \ldots$$
$$\ldots \cdot \left(1-0.14\log_{10} N + 0.0183\log_{10}^{2} N\right) \quad (13)$$

Peebles states that this formula is accurate to within about 0.8 dB over a range of about 1 to 100 for N, 0.5 to 0.999 for P_D, and 10^{-10} to 10^{-2} for P_{FA}. An approximation for the integration gain exponent can also be written directly from Eq. (13); it is

$$\alpha = 0.679\left(1+0.253P_D\right)\left(1+\frac{\log_{10}\left(1/P_{FA}\right)}{46.6}\right)\left(1-0.14\log_{10} N + 0.0183\log_{10}^{2} N\right) \quad (14)$$

6 Comparison of Calculations of Noncoherent Integration Gain

Figure 2 compares G_{nc} for the case of $P_{FA} = 10^{-8}$ and P_D = 0.5. 0.8, or 0.95 as computed using the exact results of Eqs. (10) and (11) with Eq. (9); Albersheim's equation (12) with (9); and Peebles' approximation (13). Figure 3 repeats the same conditions but with $P_{FA} = 10^{-4}$. Clearly all three methods produce generally similar estimates of G_{nc}. It can also be seen that G_{nc} is only a weak function of P_D and P_{FA}, since the curves do not vary much with either of these.

Estimates of G_{nc} based on Albersheim's or Peebles' equations are much easier to deal with because they involve simple, closed-form expressions, while computation of Marcum's Q function $Q_M(\cdot,\cdot)$ requires iterative procedures that may exhibit numerical problems for some parameters [7]. Figure 4 shows the difference in G_{nc} estimated using Albersheim's or Peebles' equations vs. that computed using the exact results of Eqs. (10) and (11) for $P_{FA} = 10^{-10}$ and three values of P_D. This particular case provides some of the larger errors for likely values of P_D and P_{FA}. The errors remain well within the ±1 dB range for both approximations over a wide range of P_D and P_{FA}.

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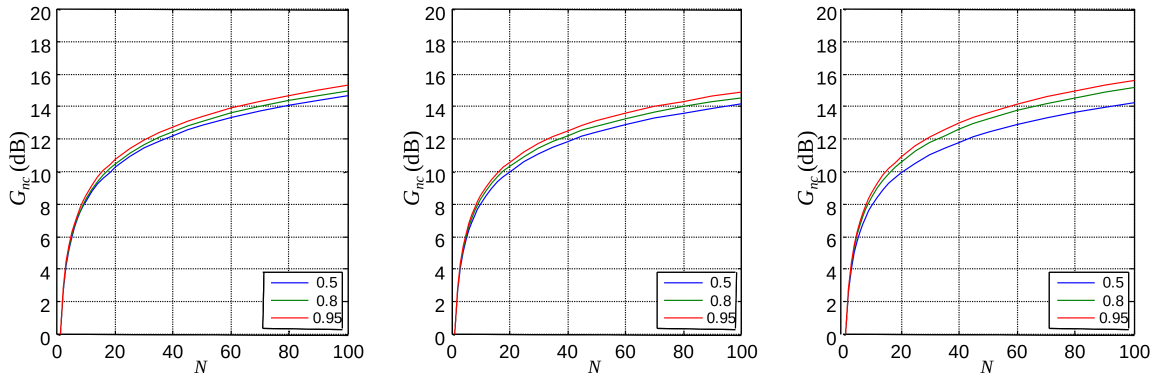
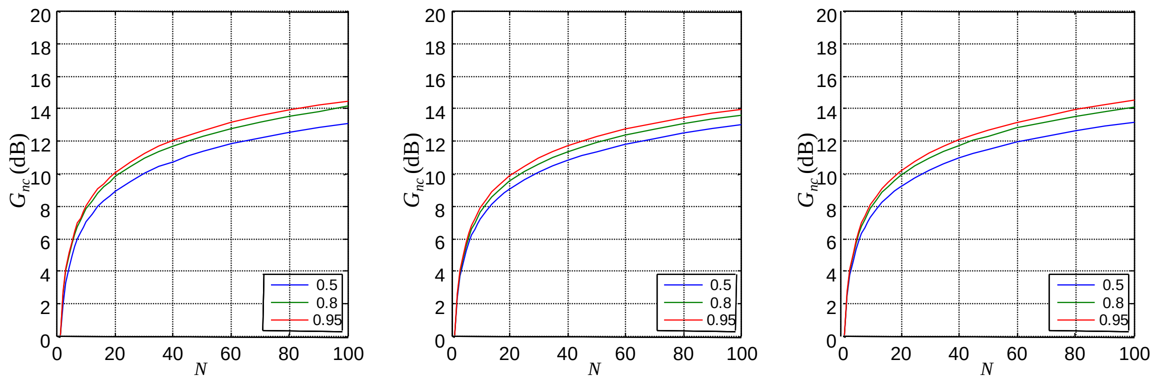


(a) (b) (c)

Figure 2. Noncoherent integration gain for $P_{FA} = 10^{-8}$ and three values of P_D. (a) Exact result. (b) Computed using Albersheim's equation. (c) Peebles' approximation.



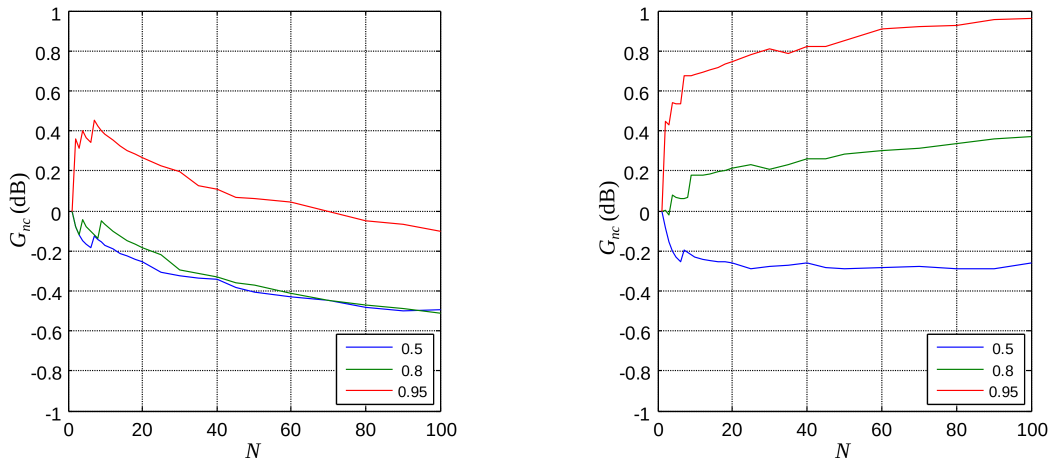
(a) (b) (c)

Figure 3. Noncoherent integration gain for $P_{FA} = 10^{-4}$ and three values of P_D. (a) Exact result. (b) Computed using Albersheim's equation. (c) Peebles' approximation.

7 Efficiency of Noncoherent Integration for Nonfluctuating Targets

Because coherent integration would reach a factor of 10 dB for $N = 10$ and 20 dB for $N = 100$, it is also clear that G_{nc} for a nonfluctuating target is less efficient than coherent integration.[1] That is, if G_{nc} is expressed in the form N^{α}, then $\alpha < 1$. (Recall that $G_c = N$, so $\alpha = 1$ for coherent integration.) In the example given earlier, $\alpha = 0.833$.

[1] This is also true for noncoherent integration with fluctuating targets; we just don't demonstrate it here.



(a) (b)

Figure 4. Error in estimate of G_{nc} as compared to that calculated using Eqs. (10) and (11) for $P_{FA} = 10^{-10}$ and three values of P_D. (a) Error in estimate using Albersheim's Eq. (12). (b) Error using Peebles' Eq. (13).

The integration exponent α can be obtained from estimates of G_{nc} by the formula

$$\alpha = \frac{\log_{10} G_{nc}}{\log_{10}(N)} = \frac{\log_{10}\left(\chi_1/\chi_{1,N}\right)}{\log_{10}(N)} \tag{15}$$

Figure 5 illustrates the efficiency of noncoherent integration for one example, a nonfluctuating target with $P_D = 0.9$ and $P_{FA} = 10^{-6}$. The exponent α exceeds 0.8 for small N, and falls to a little more than 0.7 for $N = 100$. However, over this range of N, α easily exceeds 0.5, corresponding to a $\sqrt{N}$ factor.

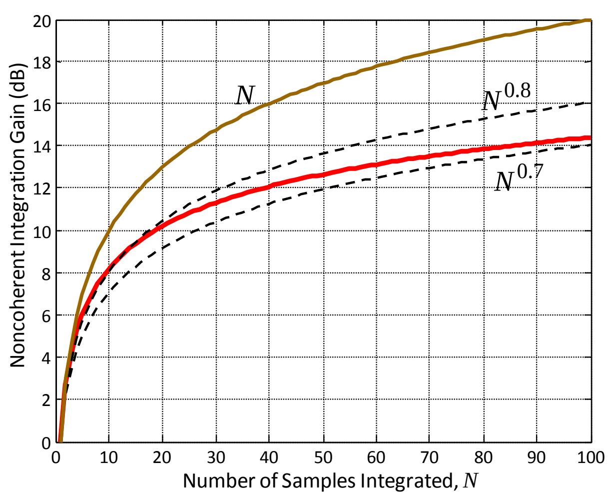


Figure 5. Noncoherent integration gain for a nonfluctuating target with $P_D = 0.9$ and $P_{FA} = 10^{-6}$, computed using Albersheim's equation.

In coherent integration, the integration factor is N independent of the SNR of the individual data samples. Figure 5 suggests that, as more samples are noncoherently integrated, the integration efficiency degrades. That is, α is relatively high for the first few samples integrated, but there are some diminishing returns as N is increased. At the same time, for a fixed P_D and P_{FA}, a larger value of N implies that the individual samples can start with a lower value of SNR, $\chi_{1,N}$. These observations suggest that noncoherent integration is more efficient when the single-sample SNR is high than when it is low.

Figure 6 plots α vs. $\chi_{1,N}$ to make this behavior explicit. When the single-sample SNR is high to begin with, very few samples need be integrated to achieve the desired detection performance, and the noncoherent integration efficiency is in the vicinity of 0.9. On the other hand, if the single-sample SNR is very low to begin with, noncoherent integration efficiency asymptotically approaches $\sqrt{N}$ ($\alpha = 0.5$). However, this occurs only for very extremely (and unrealistically) large numbers of samples integrated; the case of $\chi_{1,N} = -30$ dB and $P_{FA} = 10^{-6}$ in Figure 6, which gives $\alpha \approx 0.57$, corresponds to $N \approx 36$ million![2]

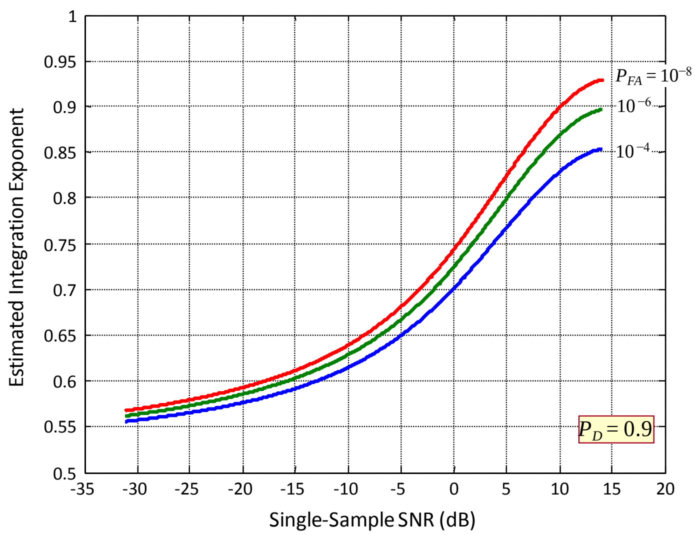


Figure 6. Noncoherent integration gain vs. single-sample SNR $\chi_{1,N}$ for nonfluctuating target with P_D = 0.9 and $P_{FA} = 10^{-4}$, 10^{-6}, and 10^{-8}, computed using Albersheim's equation.

[2] These extreme cases should be considered with some caution. They were computed using Albersheim's equation, but with parameters well out of the range over which the accuracy of Albersheim's equation is guaranteed. The exact equations using Marcum's Q function were not used because it is difficult to evaluate for such large N [7]. The trends shown above are believed to be correct, but the accuracy of the precise values is uncertain for single sample SNRs less than about -2 dB.

8 Approximations to Noncoherent Integration Gain for Detection of Fluctuating Targets

Shnidman has given useful empirical approximations for calculating single-sample SNR given P_D, P_{FA}, and N for fluctuating targets [8]. These equations play the same role for fluctuating targets as does Albersheim's equation for nonfluctuating targets. Consequently, the same strategy described above with Albersheim's equation can be applied using Shnidman's equation to estimate noncoherent integration gain for fluctuating targets. The results can be compared with more exact calculation using the equations from [1] or [2] with the software from [3] to estimate the accuracy of the procedure based on Shnidman's equations.

9 References

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