



Tracking radar targets with multiple reflection points



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Version History

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Feb 6, 2018		Version 1.2 Updated configuration parameters description
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1. Introduction

1.1. Tracking Module

In Radar Processing stack, the tracking algorithms are the implementations of the localization processing Layer. Tracker is expected to work on the Detection layer inputs, and provide localization information to the classification layers.

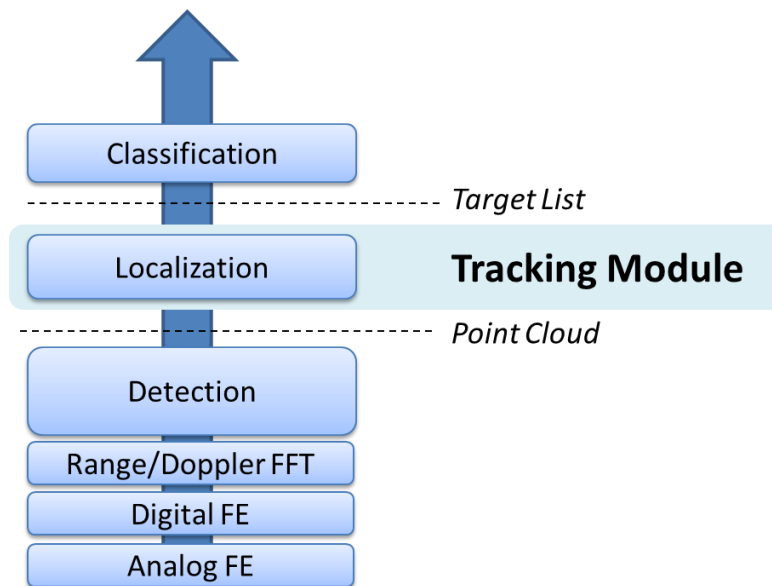


Figure 1. Radar Processing Layers

With high resolution Radar sensors, the detection layer is capable of sensing multiple reflections from real life targets, delivering a rich set of measurement vectors (in some cases, thousands of vectors per frame), known as a *Point Cloud*. Each measurement vector represents a reflection point, with Range, Azimuth, and Radial velocity. Each measurement vector may also contain reliability information.

Tracking Layer is expected to input the point cloud data, perform target localization, and report the results (a *Target List*) to a classification layer. Therefore, the output of the tracker is a set of trackable objects with certain properties (like position, velocity, physical dimensions, point density, and other features) that can be used by a classifier to make an identification decision.

1.2. Radar Geometry

The picture below illustrates single reflection point at time n . Real life radar targets are represented by multiple reflection points. Each point is represented by range, angle, and radial velocity (range rate):

- Range r , $R_{min} < r < R_{max}$
- Azimuth angle φ , $-\varphi_{max} < \varphi < +\varphi_{max}$
- Radial velocity \dot{r}

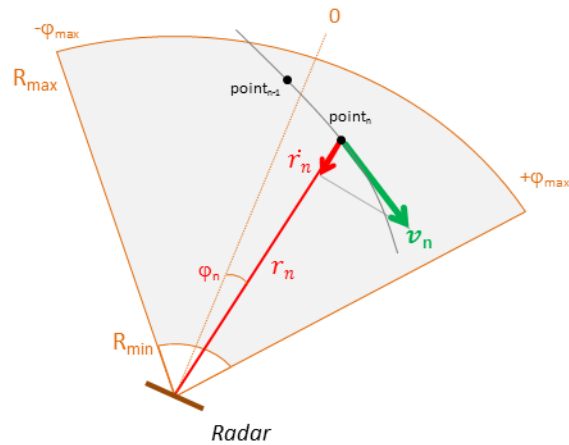


Figure 2. Radar Geometry in 2D

1.3. Choice of Tracking Coordinate System

For convenience of target motion extrapolation, we chose tracking in Cartesian coordinates. This allows for simple Newtonian linear prediction models. We chose to keep measurement inputs in polar coordinates to avoid error coupling. We will use extended Kalman filter to linearize the dependencies between tracking states and measurement vector.

Tracking can operate in either 2D or 3D Cartesian spaces. For each space, we use either constant velocity model or constant acceleration model.

1.4. 2D Space Geometry

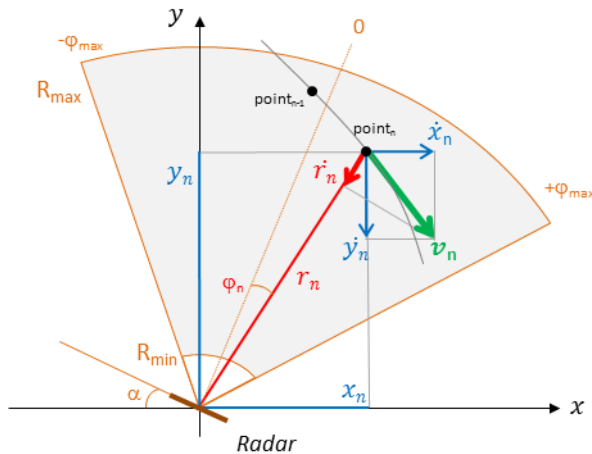


Figure 3. Tracking in 2D

The angular location coordinates are converted to Cartesian coordinates using

$$x = r \cos\left(\frac{\pi}{2} - (\alpha + \varphi)\right) = r \sin(\alpha + \varphi)$$
$$y = r \sin\left(\frac{\pi}{2} - (\alpha + \varphi)\right) = r \cos(\alpha + \varphi)$$

The objective is to track the location of the objects using the noisy measurements of range, angle, and Doppler (radial velocity)

1.5. 2D Space, Constant Velocity Model

We use Kalman filter to “refine” the location estimates. The state of the Kalman filter at time instant n is defined as

$$\mathbf{s}(n) = \mathbf{F}\mathbf{s}(n-1) + \mathbf{w}(n) \quad (1-1)$$

where the state vector $\mathbf{s}(n)$ is defined in Cartesian coordinates,

$$\mathbf{s}(n) \triangleq [x(n) \quad y(n) \quad \dot{x}(n) \quad \dot{y}(n)]^T, \quad (1-2)$$

\mathbf{F} is a transition matrix

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1-3)$$

and $\mathbf{w}(n)$ is the vector of process noise with covariance matrix $\mathbf{Q}(n)$ of size 4×4 .

The input measurement vector $\mathbf{u}(n)$ includes range, angle and radial velocity

$$\mathbf{u}(n) = [r(n) \quad \varphi(n) \quad \dot{r}(n)]^T \quad (1-4)$$

The relationship between the state of the Kalman filter and measurement vector is expressed as:

$$\mathbf{u}(n) = \mathbf{H}(\mathbf{s}(n)) + \mathbf{v}(n), \quad (1-5)$$

Where, \mathbf{H} is a measurement matrix,

$$\mathbf{H}(\mathbf{s}(n)) = \begin{bmatrix} \sqrt{x^2 + y^2} \\ \tan^{-1}(x, y) - \alpha \\ \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} \end{bmatrix}, \quad (1-6)$$

the function $\tan^{-1}(x, y)$ is defined as

$$\tan^{-1}(x, y) \triangleq \begin{cases} \tan^{-1}\left(\frac{x}{y}\right), & y > 0, \\ \frac{\pi}{2}, & y = 0, \\ \tan^{-1}\left(\frac{x}{y}\right) + \pi, & y < 0. \end{cases} \quad (1-7)$$

and $\mathbf{v}(n)$ is vector of measurement noise with covariance matrix $\mathbf{R}(n)$ of size 3×3 .

In above formulation, the measurement vector $\mathbf{u}(n)$ is related to the state vector $\mathbf{s}(n)$ via a non-linear relation. Because of this, we use Extended Kalman filter (EKF), which simplifies the relation between $\mathbf{u}(n)$ and $\mathbf{s}(n)$ by retaining only the first term in the Taylor series expansion of $\mathbf{H}(\cdot)$.

$$\mathbf{u}(n) = \mathbf{H}(\mathbf{s}_{apr}(n)) + \mathbf{J}_H(\mathbf{s}_{apr}(n))[\mathbf{s}(n) - \mathbf{s}_{apr}(n)] + \mathbf{v}(n), \quad (1-8)$$

where, $\mathbf{s}_{apr}(n)$ is a-priori estimation of state vector at time n based on $n-1$ measurements,

$$\mathbf{J}_H(\mathbf{s}) = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial \dot{x}} & \frac{\partial r}{\partial \dot{y}} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial \dot{x}} & \frac{\partial \varphi}{\partial \dot{y}} \\ \frac{\partial \dot{r}}{\partial x} & \frac{\partial \dot{r}}{\partial y} & \frac{\partial \dot{r}}{\partial \dot{x}} & \frac{\partial \dot{r}}{\partial \dot{y}} \end{bmatrix}. \quad (1-9)$$

Calculating partial derivatives (see the Appendix below):

$$\mathbf{J}_H(\mathbf{s}) = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 & 0 \\ \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} & 0 & 0 \\ \frac{y(\dot{x}y-\dot{y}x)}{(x^2+y^2)^{3/2}} & \frac{x(\dot{y}x-\dot{x}y)}{(x^2+y^2)^{3/2}} & \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix} \quad (1-10)$$

1.6. 2D Space, Constant Acceleration Model

For constant acceleration model, the state of the Kalman filter at time instant n is defined as

$$\mathbf{s}(n) = \mathbf{F}\mathbf{s}(n-1) + \mathbf{w}(n), \quad (1-11)$$

where the state vector $\mathbf{s}(n)$ is defined in Cartesian coordinates,

$$\mathbf{s}(n) = [x(n) \quad y(n) \quad \dot{x}(n) \quad \dot{y}(n) \quad \ddot{x}(n) \quad \ddot{y}(n)], \quad (1-12)$$

\mathbf{F} is a transition matrix

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & T & 0 & 0.5T^2 & 0 \\ 0 & 1 & 0 & T & 0 & 0.5T^2 \\ 0 & 0 & 1 & 0 & T & 0 \\ 0 & 0 & 0 & 1 & 0 & T \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (1-13)$$

and $\mathbf{w}(n)$ is the vector of process noise with covariance matrix $\mathbf{Q}(n)$ of size 6×6 .

The input measurement vector $\mathbf{u}(n)$ is the same as in constant velocity model, includes range, angle and radial velocity

$$\mathbf{u}(n) = [r(n) \quad \varphi(n) \quad \dot{r}(n)]^T. \quad (1-14)$$

The relationship between the state of the Kalman filter and measurement vector is expressed as:

$$\mathbf{u}(n) = \mathbf{H}(\mathbf{s}(n)) + \mathbf{v}(n), \quad (1-15)$$

where

$$\mathbf{H}(\mathbf{s}(n)) = \begin{bmatrix} \sqrt{x^2 + y^2} \\ \tan^{-1}(x, y) \\ \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} \end{bmatrix} \quad (1-16)$$

To linearize state to measurement relations, we perform partial derivatives

$$\mathbf{J}_H(\mathbf{s}) = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial \dot{x}} & \frac{\partial r}{\partial \dot{y}} & \frac{\partial r}{\partial \ddot{x}} & \frac{\partial r}{\partial \ddot{y}} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial \dot{x}} & \frac{\partial \varphi}{\partial \dot{y}} & \frac{\partial \varphi}{\partial \ddot{x}} & \frac{\partial \varphi}{\partial \ddot{y}} \\ \frac{\partial \dot{r}}{\partial x} & \frac{\partial \dot{r}}{\partial y} & \frac{\partial \dot{r}}{\partial \dot{x}} & \frac{\partial \dot{r}}{\partial \dot{y}} & \frac{\partial \dot{r}}{\partial \ddot{x}} & \frac{\partial \dot{r}}{\partial \ddot{y}} \end{bmatrix} \quad (1-17)$$

$$J_H(\mathbf{s}) = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 & 0 & 0 & 0 \\ \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} & 0 & 0 & 0 & 0 \\ \frac{y(\dot{x}y - y\dot{x})}{(x^2+y^2)^{3/2}} & \frac{x(\dot{y}x - \dot{x}y)}{(x^2+y^2)^{3/2}} & \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 & 0 \end{bmatrix} \quad (1-18)$$

1.7. 3D Space, Geometry

In the picture below, sensor is positioned at the origin. Target trajectory is shown at times $n - 2$, $n - 1$, and n . Target is moving with velocity vector \mathbf{v} . Measurement vector \mathbf{u} at time n includes range, azimuth and elevation and radial velocity:

$$\mathbf{u}(n) = [r(n) \quad \varphi(n) \quad \theta(n) \quad \dot{r}(n)]^T \quad (1-19)$$

State vector in Cartesian coordinates will be \mathbf{s}_{3DV} for constant velocity model, and \mathbf{s}_{3DA} for constant acceleration model:

$$\mathbf{s}_{3DV}(n) \triangleq [x(n) \quad y(n) \quad z(n) \quad \dot{x}(n) \quad \dot{y}(n) \quad \dot{z}(n)]^T \quad (1-20)$$

$$\mathbf{s}_{3DA}(n) \triangleq [x(n) \quad y(n) \quad z(n) \quad \dot{x}(n) \quad \dot{y}(n) \quad \dot{z}(n) \quad \ddot{x}(n) \quad \ddot{y}(n) \quad \ddot{z}(n)]^T \quad (1-21)$$

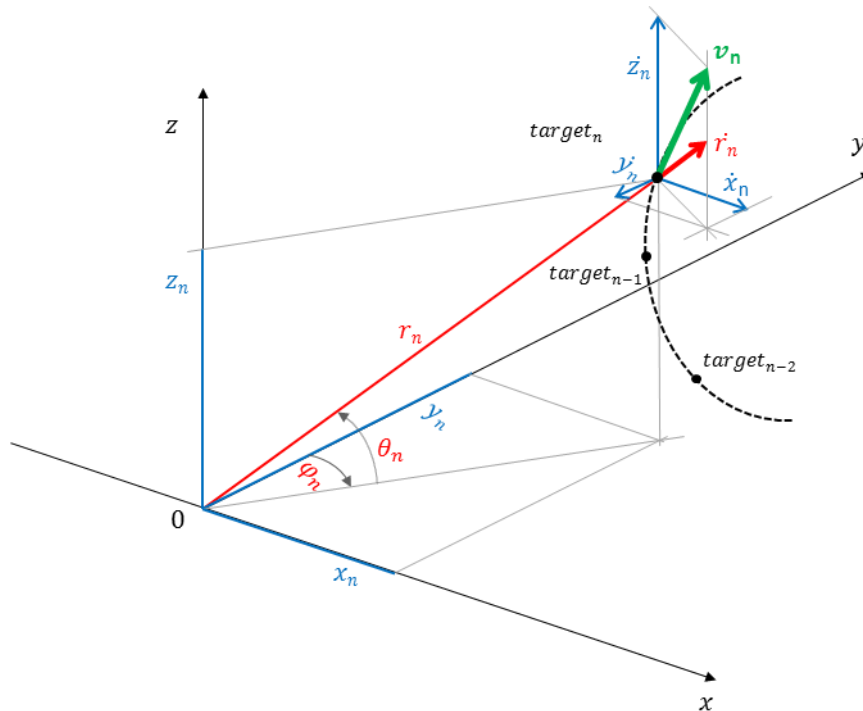


Figure 4. Tracking in 3D

1.8. 3D Space, Constant Velocity Model (3DV) and Constant Acceleration (3DA) models

We use Kalman filter to “refine” the location estimates. The state of the Kalman filter at time instant n is defined as

$$\mathbf{s}(n) = \mathbf{F}\mathbf{s}(n-1) + \mathbf{w}(n) \quad (1-22)$$

where the state vector $\mathbf{s}(n)$ is defined in Cartesian coordinates. For 3D space, we use \mathbf{s}_{3DV} or \mathbf{s}_{3DA} as defined in previous paragraph.

Transition matrix \mathbf{F} is

$$\mathbf{F}_{3DV} = \begin{bmatrix} 1 & 0 & 0 & T & 0 & 0 \\ 0 & 1 & 0 & 0 & T & 0 \\ 0 & 0 & 1 & 0 & 0 & T \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (1-23)$$

or

$$\mathbf{F}_{3DA} = \begin{bmatrix} 1 & 0 & 0 & T & 0 & 0 & 0.5T^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & T & 0 & 0 & 0.5T^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & T & 0 & 0 & 0.5T^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & T & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & T \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (1-24)$$

and $\mathbf{w}(n)$ is the vector of process noise with covariance matrix $\mathbf{Q}(n)$ of size 6×6 or 9×9.

The input measurement vector $\mathbf{u}(n)$ includes range, azimuth, elevation, and radial velocity

$$\mathbf{u}(n) = [r(n) \quad \varphi(n) \quad \theta(n) \quad \dot{r}(n)]^T \quad (1-25)$$

The relationship between the state of the Kalman filter and measurement vector is expressed as:

$$\mathbf{u}(n) = \mathbf{H}(\mathbf{s}(n)) + \mathbf{v}(n), \quad (1-26)$$

Where, \mathbf{H} is a measurement matrix,

$$\mathbf{H}(\mathbf{s}(n)) = \begin{bmatrix} \sqrt{x^2 + y^2 + z^2} \\ \tan^{-1}(x, y) \\ \tan^{-1}\left(z, \sqrt{x^2 + y^2}\right) \\ \frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2 + y^2 + z^2}} \end{bmatrix}, \quad (1-27)$$

the function $\tan^{-1}(a, b)$ is defined as

$$\tan^{-1}(a, b) \triangleq \begin{cases} \tan^{-1}\left(\frac{a}{b}\right), & b > 0, \\ \frac{\pi}{2}, & b = 0, \\ \tan^{-1}\left(\frac{a}{b}\right) + \pi, & b < 0. \end{cases} \quad (1-28)$$

and $\mathbf{v}(n)$ is vector of measurement noise with covariance matrix $\mathbf{R}(n)$ of size 4×4 .

In above formulation, the measurement vector $\mathbf{u}(n)$ is related to the state vector $\mathbf{s}(n)$ via a non-linear relation. Because of this, we use Extended Kalman filter (EKF), which simplifies the relation between $\mathbf{u}(n)$ and $\mathbf{s}(n)$ by retaining only the first term in the Taylor series expansion of $\mathbf{H}(\cdot)$.

$$\mathbf{u}(n) = \mathbf{H}(\mathbf{s}_{apr}(n)) + \mathbf{J}_H(\mathbf{s}_{apr}(n))[\mathbf{s}(n) - \mathbf{s}_{apr}(n)] + \mathbf{v}(n), \quad (1-29)$$

where, $\mathbf{s}_{apr}(n)$ is a-priori estimation of state vector at time n based on $n-1$ measurements,

$$\mathbf{J}_H(\mathbf{s}) = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} & \frac{\partial r}{\partial \dot{x}} & \frac{\partial r}{\partial \dot{y}} & \frac{\partial r}{\partial \dot{z}} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} & \frac{\partial \varphi}{\partial \dot{x}} & \frac{\partial \varphi}{\partial \dot{y}} & \frac{\partial \varphi}{\partial \dot{z}} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} & \frac{\partial \theta}{\partial \dot{x}} & \frac{\partial \theta}{\partial \dot{y}} & \frac{\partial \theta}{\partial \dot{z}} \\ \frac{\partial \dot{r}}{\partial x} & \frac{\partial \dot{r}}{\partial y} & \frac{\partial \dot{r}}{\partial z} & \frac{\partial \dot{r}}{\partial \dot{x}} & \frac{\partial \dot{r}}{\partial \dot{y}} & \frac{\partial \dot{r}}{\partial \dot{z}} \end{bmatrix}. \quad (1-30)$$

Calculating partial derivatives (see the Appendix below):

$$\mathbf{J}_H(\mathbf{s}_{3DV}) = \begin{bmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} & 0 & 0 & 0 \\ \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} & 0 & 0 & 0 & 0 \\ -\frac{x}{r^2} \frac{z}{\sqrt{x^2+y^2}} & -\frac{y}{r^2} \frac{z}{\sqrt{x^2+y^2}} & \frac{\sqrt{x^2+y^2}}{r^2} & 0 & 0 & 0 \\ \frac{y(\dot{x}y - \dot{y}x) + z(\dot{x}z - \dot{z}x)}{r^3} & \frac{x(\dot{y}x - \dot{x}y) + z(\dot{y}z - \dot{z}y)}{r^3} & \frac{x(\dot{z}x - \dot{x}z) + y(\dot{z}y - \dot{y}z)}{r^3} & \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{bmatrix} \quad (1-31)$$

$$J_H(S_{3DA}) = \begin{bmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{y}{x^2+y^2} & -\frac{x}{x^2+y^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{x}{r^2} \frac{z}{\sqrt{x^2+y^2}} & -\frac{y}{r^2} \frac{z}{\sqrt{x^2+y^2}} & \frac{\sqrt{x^2+y^2}}{r^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{y(\dot{x}y-\dot{y}x)+z(\dot{x}z-\dot{z}x)}{r^3} & \frac{x(\dot{y}x-\dot{x}y)+z(\dot{y}z-\dot{z}y)}{r^3} & \frac{x(\dot{z}x-\dot{x}z)+y(\dot{z}y-\dot{y}z)}{r^3} & \frac{x}{r} & \frac{y}{r} & \frac{z}{r} & 0 & 0 & 0 \end{bmatrix} \quad (1-32)$$

2. Kalman Filter Operations

The notation we use:

- $s_i(n)$ – State vector of tracking object i at time n . Each tracking object has its own state vector, which is predicted and updated independently. For simplicity reasons we omit the tracking index.
- $s_{apr}(n)$ – A-priori (predicted) estimates of tracking state at time n .
- $\mathbf{P}(n)$ – State vector estimation error covariance matrix at time n , defined as $\mathbf{P}(n) = \text{Cov}[s(n) - s_{apr}(n)]$.
- $\mathbf{P}_{apr}(n)$ – A-priori (predicted) estimates of state vector covariance matrix at time n .

2.1. Prediction Step

With available measurement until time instant $n-1$, the a priori state and error covariance estimates are obtained using:

$$s_{apr}(n) = \mathbf{F}s(n-1) \quad (2-1)$$

$$\mathbf{P}_{apr}(n) = \mathbf{F}\mathbf{P}(n-1)\mathbf{F}^T + \mathbf{Q}(n-1). \quad (2-2)$$

The above equations constitute the *prediction* step of the Kalman filter. $\mathbf{Q}(n)$ is the process noise covariance matrix. See section 2.3 for the details.

In addition, we also compute $\mathbf{H}(s_{apr}(n))$ using formula above.

2.2. Update Step

As measurements at time instant n become available, state and error covariance estimates are updated in the following measurement update procedure:

- a) Compute innovation (or measurement residual)

$$\mathbf{y}(n) = \mathbf{u}(n) - \mathbf{H}(s_{apr}(n)) \quad (2-3)$$

- b) Compute innovation covariance

$$\mathbf{C}(n) = \mathbf{J}_H(s_{apr}(n))\mathbf{P}_{apr}(n)\mathbf{J}_H^T(s_{apr}(n)) + \mathbf{R}(n) \quad (2-4)$$

- c) Compute Kalman gain

$$\mathbf{K}(n) = \mathbf{P}_{apr}(n)\mathbf{J}_H^T(s_{apr}(n)) \text{inv}[\mathbf{C}(n)] \quad (2-5)$$

d) Compute a-posteriori state vector

$$\mathbf{s}(n) = \mathbf{s}_{apr}(n) + \mathbf{K}(n)\mathbf{y}(n) \quad (2-6)$$

e) Compute a-posteriori error covariance

$$\mathbf{P}(n) = \mathbf{P}_{apr}(n) - \mathbf{K}(n)\mathbf{J}_H(\mathbf{s}_{apr}(n))\mathbf{P}_{apr}(n) \quad (2-7)$$

2.3. Design of Process Noise Matrix

This section is adopted from the works in [1].

The choice of $\mathbf{Q}(n)$ is important for the behavior of the Kalman filter. If \mathbf{Q} is too small then the filter will be overconfident in its prediction model and will diverge from the actual solution. If \mathbf{Q} is too large than the filter will be too much influenced by the noise in the measurements and perform sub-optimally.

The kinematic system (a system that can be modeled using Newton's equations of motion) is continuous, i.e. their inputs and outputs can vary at any arbitrary point in time. However, Kalman filters used here are discrete. We sample the system at regular intervals. Therefore we must find the discrete representation for the noise term in the equation above. This depends on what assumptions we make about the behavior of the noise. We will consider two different models for the noise.

2.3.1. Continuous White Noise Model

Let's say that we need to model the position, velocity, and acceleration. We can then assume that acceleration is constant for each discrete time step. Of course, there is process noise in the system and so the acceleration is not actually constant. The tracked object will alter the acceleration over time due to external, un-modeled forces. In this section we will assume that the acceleration changes by a continuous time zero-mean white noise.

Since the noise is changing continuously we will need to integrate to get the discrete noise for the discretization interval that we have chosen. We will not prove it here, but the equation for the discretization of the noise is

$$\mathbf{Q} = \int_0^{\Delta t} \mathbf{F}(t)\mathbf{Q}_c\mathbf{F}^T(t)dt, \quad (2-8)$$

where \mathbf{Q}_c is the continuous noise. The general reasoning should be clear. $\mathbf{F}(t)\mathbf{Q}_c\mathbf{F}^T(t)$ is a projection of the continuous noise based on our process model $\mathbf{F}(t)$ at the instant t . We want to know how much noise is added to the system over a discrete interval Δt , so we integrate this expression over the interval $[0, \Delta t]$.

For the second order Newtonian system, the fundamental matrix is

$$\mathbf{F} = \begin{bmatrix} 1 & \Delta t & \Delta t^2/2 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix}. \quad (2-9)$$

We now define the continuous noise as

$$\mathbf{Q}_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Phi_s \quad (2-10)$$

where Φ_s is the spectral density of the white noise. This can be derived, but is beyond the scope for now. In practice we often do not know the spectral density of the noise, and so this turns into an "engineering" factor - a number we experimentally tune until our filter performs as we expect. We can see that the matrix that Φ_s is multiplied by effectively assigns the power spectral density to the acceleration term. This makes sense; we assume that the system has constant acceleration except for the variations caused by noise. The noise alters the acceleration.

Computing the integral, we obtain

$$\mathbf{Q} = \begin{bmatrix} \frac{\Delta t^5}{20} & \frac{\Delta t^4}{8} & \frac{\Delta t^3}{6} \\ \frac{\Delta t^4}{8} & \frac{\Delta t^3}{3} & \frac{\Delta t^2}{2} \\ \frac{\Delta t^3}{6} & \frac{\Delta t^2}{2} & \Delta t \end{bmatrix} \Phi_s \quad (2-11)$$

Extrapolating back to 6 states,

$$\mathbf{Q} = \begin{bmatrix} \frac{\Delta t^5}{20} & 0 & \frac{\Delta t^4}{8} & 0 & \frac{\Delta t^3}{6} & 0 \\ 0 & \frac{\Delta t^5}{20} & 0 & \frac{\Delta t^4}{8} & 0 & \frac{\Delta t^3}{6} \\ \frac{\Delta t^4}{8} & 0 & \frac{\Delta t^3}{3} & 0 & \frac{\Delta t^2}{2} & 0 \\ 0 & \frac{\Delta t^4}{8} & 0 & \frac{\Delta t^3}{3} & 0 & \frac{\Delta t^2}{2} \\ \frac{\Delta t^3}{6} & 0 & \frac{\Delta t^2}{2} & 0 & \Delta t & 0 \\ 0 & \frac{\Delta t^3}{6} & 0 & \frac{\Delta t^2}{2} & 0 & \Delta t \end{bmatrix} \Phi_s \quad (2-12)$$

2.3.2. Piecewise White Noise Model

Another model for the noise assumes that the that highest order term (say, acceleration) is constant for the duration of each time period, but differs for each time period, and each of these is uncorrelated between time periods. In other words there is a discontinuous jump in acceleration at each time step. This is subtly different than the model above, where we assumed that the last term had a continuously varying noisy signal applied to it.

We will model this as

$$f(x) = \mathbf{F}x + \mathbf{\Gamma}w \quad (2-13)$$

where $\mathbf{\Gamma}$ is the noise gain of the system, and w is the constant piecewise acceleration (or velocity, or jerk, etc).

For the second order system

$$\mathbf{F} = \begin{bmatrix} 1 & \Delta t & \frac{\Delta t^2}{2} \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix} \quad (2-14)$$

In one time period, the change in acceleration will be $w(t)$, change in velocity will be $w(t)\Delta t$, and change in position will be $w(t)\Delta t^2/2$. This gives us

$$\mathbf{\Gamma} = \begin{bmatrix} \Delta t^2/2 \\ \Delta t \\ 1 \end{bmatrix} \quad (2-15)$$

The covariance of the process noise is then

$$\mathbf{Q} = E[\mathbf{\Gamma}w(t)w(t)\mathbf{\Gamma}^T] = \mathbf{\Gamma}\sigma_w^2\mathbf{\Gamma}^T \quad (2-16)$$

$$\mathbf{Q} = \begin{bmatrix} \frac{\Delta t^4}{4} & \frac{\Delta t^3}{2} & \frac{\Delta t^2}{2} \\ \frac{\Delta t^3}{2} & \Delta t^2 & \Delta t \\ \frac{\Delta t^2}{2} & \Delta t & 1 \end{bmatrix} \sigma_w^2 \quad (2-17)$$

It is not clear whether this model is more or less correct than the continuous model - both are approximations to what is happening to the actual object. Only experience and experiments can guide to the appropriate model. It is expected that either model provides reasonable results, but typically one will perform better than the other.

The advantage of the second model is that we can model the noise in terms of σ^2 which we can describe in terms of the motion and the amount of error we expect. The first model requires us to specify the spectral density, which is not very intuitive, but it handles varying time samples much more easily since the noise is integrated across the time period. However, these are not fixed rules - use whichever model (or a model of your own devising) based on testing how the filter performs and/or your knowledge of the behavior of the physical model.

A good rule of thumb is to set σ somewhere from $\frac{1}{2}\Delta a$ to Δa , where Δa is the maximum amount that the acceleration will change between sample periods. In practice we pick a number, run simulations on data, and choose a value that works well.

3. Group Tracking

3.1. High Level Algorithm Design

With advances in detection accuracy, the real world radar targets (cars, pedestrians, walls, landing ground, etc.) are presented to a tracking processing layer as a set of multiple reflection points. Those detection points form a group of correlated measurements with range, angle, and angular velocity. Of course, at any time there could be multiple real world targets. Therefore, we seek a tracker capable of working with multiple target groups.

The group tracking approach is illustrated in figure below.

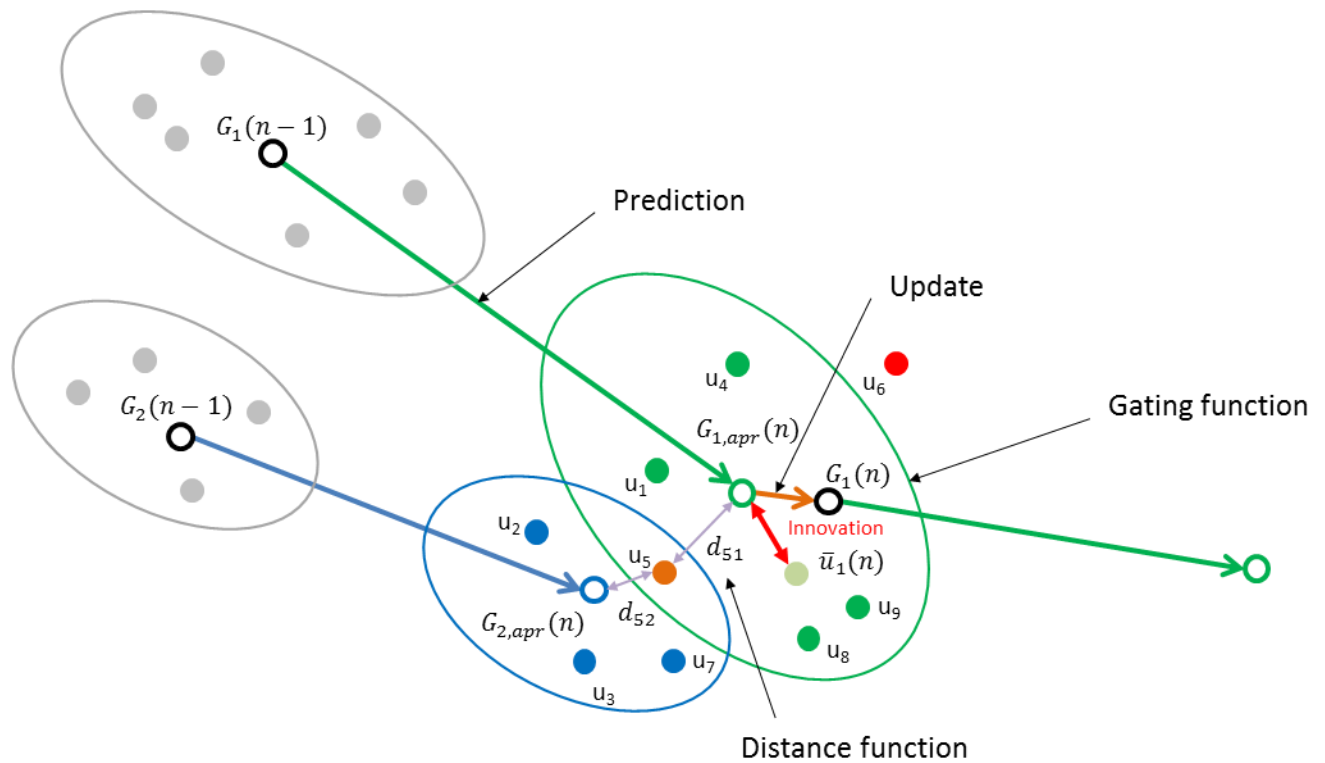


Figure 5. Group Tracking

3.2. Group Tracking Block Diagram

The algorithm flows through the major steps shown in a block diagram and described below:

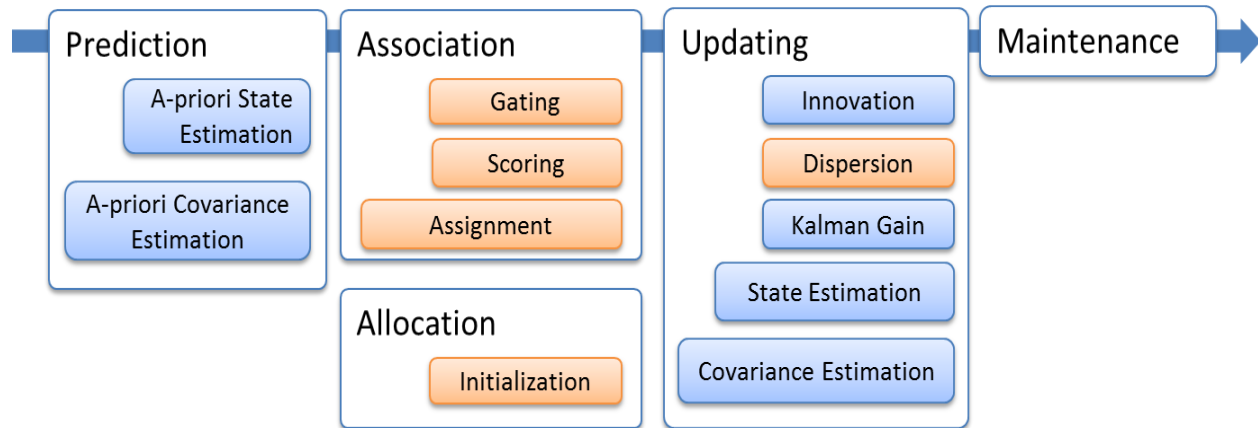


Figure 6. Tracking Block Diagram

The functions shown in blue are classical extended Kalman filter operations. The functions are shown in brown are additions to support multipoint grouping.

3.3. Prediction Step

We use the Kalman filter prediction process to estimate tracking group centroid for time n based on state and process covariance matrices estimated at time $n-1$. We compute a-priori state and covariance estimations for each trackable object. At this step we also compute measurement vector estimations. See section 2.1 for details.

3.4. Association Step

Assume existence of one or more tracks and the associated predicted state vector. For each given track we form a gate about the predicted centroid. The gate should account for a) target maneuver, for the dispersion of the group and for the measurement noise.

We use group residual covariance matrix to build an ellipsoid in 3D measurement space about the tracking group centroid. The ellipsoid will represent a *gating function* to qualify individual measurements we observe at time n . The gating function design is explained in the section below.

For the measurements within the gate, we compute *normalized distance function* as a cost function to associate a measurement to each track

The assignment process minimizes the cost function, assigning one measurement at a time to the closest track. This creates a set of measurements associated with each track

3.4.1. Gating Function

The gating function represents the amount of innovations we are willing to accept given current state of uncertainty that exists in a current track.

To measure the amount of uncertainty, we define the group residual covariance matrix as

$$\mathbf{C}_G = \mathbf{J}_H \left(\mathbf{s}_{apr}(n) \right) \mathbf{P}_{apr}(n) \mathbf{J}_H^T \left(\mathbf{s}_{apr}(n) \right) + \mathbf{R}_G + \mathbf{C}_D \quad (3-1)$$

Note that this group covariance matrix \mathbf{C}_G is between a member of the measurement group and the group centroid.

Compare group residual covariance with formula for individual target tracking.

The term $\mathbf{J}_H \left(\mathbf{s}_{apr}(n) \right) \mathbf{P}_{apr}(n) \mathbf{J}_H^T \left(\mathbf{s}_{apr}(n) \right)$ represents the uncertainty in the centroid due to target maneuvering, and is similar to the term used for individual target tracking.

The term \mathbf{R}_G is the measurement error covariance matrix. Since we are building a gate in measurements coordinates, we don't need any transformation.

Finally, the term \mathbf{C}_D is the estimation of group track dispersion matrix.

For each existing track i , for all measurement vectors j we obtained at time n , we define a distance function d_{ij}^2 , which represents the amount of innovation the new measurement adds to an existing track.

$$d_{ij}^2 \triangleq \mathbf{y}_{ij}^T \text{inv}(\mathbf{C}_i) \mathbf{y}_{ij} \quad (3-2)$$

$$d_{ij}^2 = [\mathbf{u}_j(n) - H_i(\hat{\mathbf{s}}^{-1}(n))]^T \text{inv}(\mathbf{C}_{Gi}(n)) [\mathbf{u}_j(n) - H_i(\hat{\mathbf{s}}^{-1}(n))] \quad (3-3)$$

We define the chi-squared test (because the sum of squares of M Gaussian random variables with zero mean is chi-square distribution with degree of freedom M) limits the amount of innovation we are willing to accept as

$$d_{ij}^2 < G \quad (3-4)$$

The boundary condition represents arbitrarily oriented ellipsoid centered at a-priory expectation of the measurement vector, while G represents the largest distance from to the measurement that we are going to accept into the group. However, choice of constant G had shown stability issues. Consider the picture below. Two ellipsoids in XY coordinates with same gate center and $G=16$ are shown. The larger ellipsoid represents the \mathbf{C}_G matrix with larger dispersion. Therefore, the parameter G can be thought of as a constant volume magnifying factor to the expected errors described by \mathbf{C}_G . Now, imagine the state, were the group track associates a measurement with large, but still accepted normalized distance. As a result of this "acquisition", the track becomes more dispersed, the covariance matrix grows, and the

gating ellipse grows as well, leading to more distant acquisitions. Other way around, with more compact measurements, the dispersion decreases, leading towards less and less gating volume.

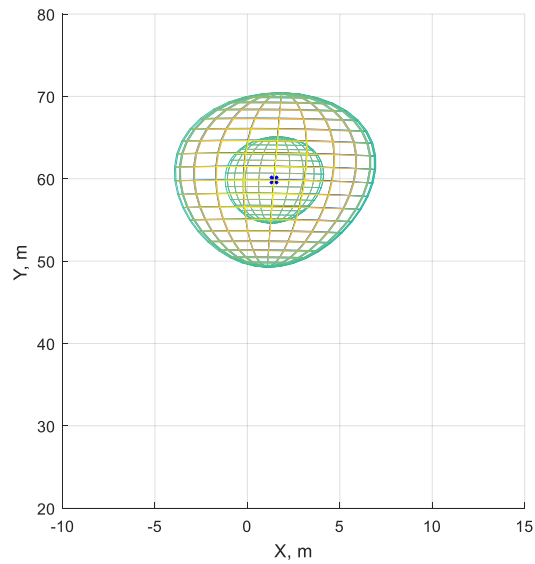


Figure 7. Ellipsoid Gating with G=16

Therefore, the proposed gating solution is based on constant volume concept. We are seeking the gating function that produces the same volume ellipsoid. Given the constant volume V we compute G . Now, the group tracks with significant amount of dispersion will see the decreasing reach, leading towards less aggressive track acquisitions. The tracks with access dispersion (those that have incompatible measurements) are going to split. That is the desired behaviour.

3.4.2. Scoring Function

The picture below illustrates the post gating situation, where measurement vectors $\{u_1, u_2, u_3, u_7\}$ pass the gating test for the green track, while vectors $\{u_3, u_4, u_5, u_8, u_9\}$ pass the gating test for the blue track.

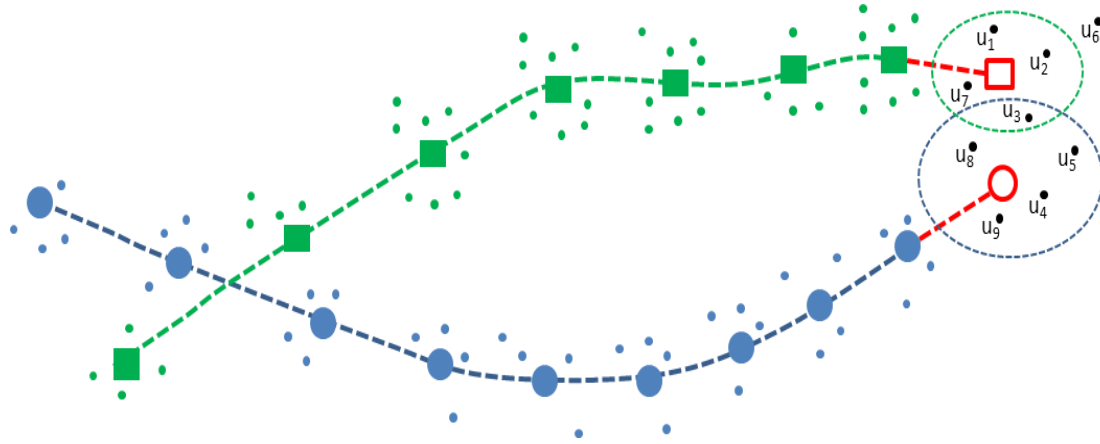


Figure 8. Scoring function Illustration

As shown in [2], the likelihood function (assuming Gaussian distribution for the residual) associated with assignment of observation j to track i is

$$g_{ij} = \frac{e^{-\frac{d_{ij}^2}{2}}}{(2\pi)^{M/2} \sqrt{|C_i|}} \quad (3-5)$$

Where, $|C_i|$ is determinant of the residual covariance matrix for track i , $d_{ij}^2 = y_{ij}^T \text{inv}(C_i) y_{ij}$, and y_{ij} is residual vector from observation j to track i .

To maximize the g_{ij} , by taking the logarithm, we derive the scoring criteria we want to minimize:

$$D_{ij}^2 = \ln|C_i| + d_{ij}^2 \quad (3-6)$$

3.5. Allocation Step

For measurements not associated with any track (that are outside of any existing gate), new group tracker is allocated and initialized. This is an iterative process, similar to DBSCAN clustering algorithm. It is significantly simpler, since it is only done for the leftover measurements.

We first select a leading measurement and set a centroid (range/angle) equal to it. The leading point's radial velocity is used to unroll other candidate's radial velocities. One candidate at a time we first check whether the point is within velocity bounds (velocity check, followed by a distance check. If passed, the centroid is recalculated, and point is added to the cluster. Once finished, we perform the few qualifying tests for cluster. We may need to see minimal number of measurements, strong enough combined SNR, and/or minimal amount of dynamicity of the centroid. If passed, we create (allocate) a new tracking object and use the associated points to initialize dispersion matrices. Clusters with fewer points are ignored.

3.6. Updating Step

This process is similar to the steps described in 2.2. Below we only outline the differences.

Tracks are updated based on the set of associated measurements computed at association step. For each track, we first compute $\bar{\mathbf{u}}(n)$ which is mean of all associated measurements.

Then, we compute the amount of innovation using the mean of all measurements associated with it:

$$\mathbf{y}(n) = \bar{\mathbf{u}}(n) - \mathbf{H}(\mathbf{s}_{apr}(n)) \quad (3-7)$$

Then, we compute residual centroid covariance matrix:

$$\mathbf{C}_c = \mathbf{J}_H(\mathbf{s}_{apr}(n)) \mathbf{P}_{apr}(n) \mathbf{J}_H^T(\mathbf{s}_{apr}(n)) + \mathbf{R}_c \quad (3-8)$$

where \mathbf{R}_c is a measurement noise covariance matrix, computed as below:

$$\mathbf{R}_c = \frac{\mathbf{R}_D}{N_A} + f(N_A, \hat{N}) \mathbf{C}_D \quad (3-9)$$

where,

\mathbf{C}_D is the estimation of group track dispersion matrix,

N_A is the number of measurements associated with given track,

\hat{N} is the estimated number of elements in target tracked by the given track,

$f(N_A, \hat{N})$ is a weighting factor that is a function of the number of observations and estimated number of elements in the target.

From the equation above, measurement noise covariance matrix is the sum of two factors. The first term $\frac{\mathbf{R}_D}{N_A}$ represents the error in measuring the centroid due to radar measurement error and is decreased by the number of the measurements associated with tracking centroid. The second term represents the uncertainty due to the fact that not all the elements may have been observed. We define the weighting factor for the case with no false detections ($N_A \leq \hat{N}$) as:

$$f(N_A, \hat{N}) = \frac{\hat{N} - N_A}{(\hat{N} - 1)N_A} \quad (3-10)$$

Note the limits on this function: $f(N_A, \hat{N}) = 0$ when all elements are detected and $f(N_A, \hat{N}) = 1$ when we received single associated measurement.

We take \hat{N} as a configuration parameter, and estimate \mathbf{C}_D recursively as described below.

For the measurement vector $\mathbf{u}(n) = [r(n) \quad \varphi(n) \quad \dot{r}(n)]^T$, the dispersion matrix is

$$\mathbf{D} = \begin{bmatrix} d_{rr}^2 & d_{r\varphi}^2 & d_{r\dot{r}}^2 \\ d_{r\varphi}^2 & d_{\varphi\varphi}^2 & d_{\varphi\dot{r}}^2 \\ d_{r\dot{r}}^2 & d_{\varphi\dot{r}}^2 & d_{\dot{r}\dot{r}}^2 \end{bmatrix} \quad (3-11)$$

Where $d_{ab}^2 \triangleq \frac{1}{N} \sum_{i=1}^N (a_i - \bar{a})(b_i - \bar{b})$, and $a, b \in \{r, \varphi, \dot{r}\}$, and \bar{a}, \bar{b} are means computed for a given group of measurements in a standard manner.

$$\mathbf{C}_D = (1 - \alpha)\mathbf{C}_D(n - 1) + \alpha\mathbf{D} \quad (3-12)$$

Once \mathbf{R}_C is computed, the rest steps (computing the Kalman gain, state and covariance matrices) are identical to in 2.2.

3.7. Maintenance Step

Each track goes through a life cycle of events. At maintenance step we may decide to change the state or to delete the track that is not used any more.

4. Implementation Details

4.1. Group Tracker

Tracking algorithm is implemented as a library. Application task creates an algorithm instance with configuration parameters that describe sensor, scenery, and behavior of radar targets. Algorithm is called once per frame from Application Task context. It is possible to create multiple instances of group tracker.

The figure below explains the steps algorithm goes during each frame call. Algorithm inputs measurement data in Polar coordinates (range, angle, Doppler), and tracks objects in Cartesian space. Therefore we use Extended Kalman Filter (EKF) process.

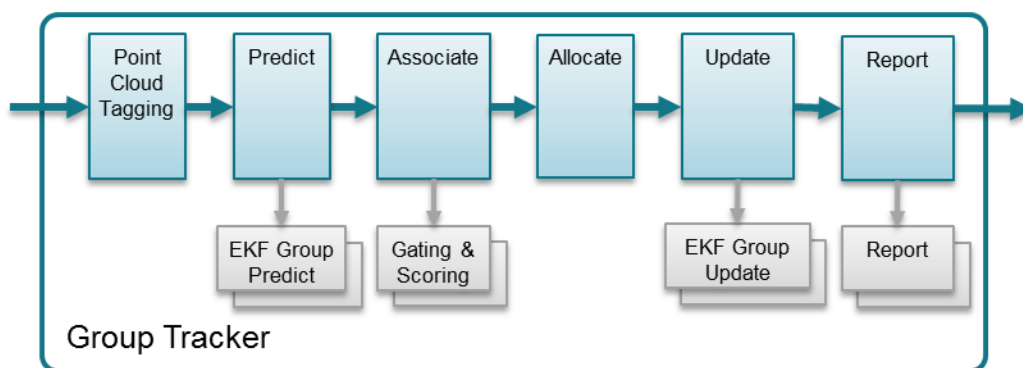


Figure 9. Group Tracking Algorithm

Point cloud input is first tagged based on scene boundaries. Some points may be get tagged as “outside the boundaries”, and will be ignored in association and allocation processes.

Predict function estimates tracking group centroid for time n based on state and process covariance matrices estimated at time $n-1$. We compute a-priori state and error covariance estimations for each trackable object. At this step we also compute measurement vector estimations.

Association function allows each tracking unit to indicate whether each measurement point is “close enough” (gating), and if it is, to provide the bidding value (scoring). Point is assigned to a highest bidder. Points not assigned, are going through an Allocate function. During the Allocation process, points are first joined into a sets based on their proximity in measurement coordinates. Each set becomes a candidate for allocation decision. It has to pass multiple tests to become a new track. Once passed, the new tracking unit is allocated.

During Update step, tracks are updated based on the set of associated points. We compute the innovation, Kalman gain, and a-posteriori state vector and error covariance. In addition to classic EKF, the error covariance calculation includes group dispersion in measurement noise covariance matrix. The Report function queries each tracking unit and produces the algorithm output.

4.2. Configuration Parameters

The configuration parameters are used to configure Tracking algorithm. They shall be adjusted to match customer usage case based on particular scenery and targets characteristics. Parameters are divided into mandatory, and optional (advanced). Mandatory parameters are described below.

4.2.1. Mandatory Configuration Parameters

Table 1. Mandatory Configuration Parameters

Parameter	Default, TM	Default, PC	Dim	Description
maxNumPoints	250	250	-	Maximum Number of Detection Points per frame
maxNumTracks	20	20	-	Maximum Number of Targets to track at any given time
stateTrackingVectorType	2DA	2DA	-	2D={x,y,vx,vy} 2DA={x,y,vx,vy,ax,ay} Currently, this is the only supported option 3D={x,y,z,vx,vy,vz} 3DA={x,y,z,vx,vy,vz,ax,ay,az}
initialRadialVelocity	-20	0	m/s	Expected target radial velocity at the moment of detection
maxRadialVelocity	N/A	N/A	m/s	Maximum absolute radial velocity reported by sensor. This shall match sensor chirp configuration
radialVelocityResolution	N/A	N/A	m/s	Minimal non-zero radial velocity reported by sensor. This shall match sensor chirp configuration
maxAccelerationX	0	2	m/s ²	Maximum targets acceleration. Used to compute processing noise matrix
maxAccelerationY	20	2	m/s ²	Maximum targets acceleration. Used to compute processing noise matrix
deltaT	N/A	N/A	ms	Frame Rate. This shall match sensor chirp configuration
verbosityLevel	NONE	NONE	-	A bit mask representing levels of verbosity: NONE WARNING DEBUG ASSOCIATION DEBUG GATE_DEBUG MATRIX_DEBUG

4.2.2. Advanced parameters

Advanced parameters are divided into few sets. Each set can be omitted, and defaults will be used by an algorithm. Customer is expected to modify needed parameters to achieve better performance.

Scenery Parameters

This set of parameters describes the scenery. It allows user to configure the tracker with expected boundaries, and areas of static behavior. User can define up to 2 boundary boxes, and up to 2 static boxes. Boxes coordinates are in meters, sensor is assumed at (0, 0) of Cartesian (X, Y) space.

Table 2. Scenery Parameters

Parameter	Default, TM	Default, PC		Description
numBoundaryBoxes	1U	0	-	Number of boundary boxes defined. Points outside boundary box will be ignored
boundaryBox[2]	{-1.f, 12.f,15.f,75.f},	{0, 0, 0, 0},	m	(left, right, bottom, top)

	{0, 0, 0, 0}	{0, 0, 0, 0}		
numStaticBoxes	1U	0	-	Number of static boxes defined. Targets inside static box are allowed to persist as static
staticBox[2]	{0.f, 11.f,19.f,50.f}, {0, 0, 0, 0}	{0, 0, 0, 0}, {0, 0, 0, 0}	m	(left, right, bottom, top)

Measurement Standard Deviation Parameters

This set of parameters is used to estimate standard deviation of the reflection point measurements.

Table 3. Measurements Standard Deviation Parameters

Parameter	Default, TM	Default, PC		Description
LengthStd	1/3.46	1/3.46	m	Expected standard deviation of measurements in target length dimension
WidthStd	1/3.46	1/3.46	m	Expected standard deviation of measurements in target width dimension
DopplerStd	1.f	1.f	m/s	Expected standard deviation of measurements of target radial velocity

Typically, the uniform distribution of reflection points across target dimensions can be assumed. In such cases, standard deviation on the interval [a,b] can be computed as below.

$$\sigma = \frac{b - a}{\sqrt{12}}$$

For example, for the targets of 1m wide, standard deviation can be configured as $\frac{1}{\sqrt{12}}$.

Allocation Parameters

The reflection points reported in point cloud are associated with existing tracking instances. Points that don't get associated are subjects for the allocation decision. Each candidate point is clustered into an allocation set. To join the set, each point needs to be within maxDistance and maxVelThre from the set's centroid. Once the set is formed, it has to have more than setPointsThre members, and pass the minimal velocity and SNR thresholds.

Table 4. Allocation Parameters

Parameter	Default, TM	Default, PC	Dim	Description
setSNRThre	-1.f	150.f	-	Minimum total SNR for the allocation set, linear sum of power ratios
setSNRObscThre	-1.f	250.f	-	Minimum total SNR for the allocation set, linear sum of power ratios, when obscured by another target
setVelThre	1.f	0.1f	m/s	Minimum radial velocity of the allocation set centroid
setPointsThre	3U	5U	-	Minimum number of points in the allocation set
maxDistanceThre	4.f	1.f	m ²	Maximum squared distance between candidate and

				centroid to be part of the allocation set
maxVelThre	2.f	2.f	m/s	Maximum velocity difference between candidate and centroid to be part of the allocation set

State Transition Parameters

Each tracking instance can be in either FREE, DETECT, or ACTIVE state. Once per frame the instance can get HIT (have non-zero points associated to a target instance) or MISS (no points associated) event.

Once in FREE state, the transition to DETECT state is made by the allocation decision. See section **Error! eference source not found.** for the allocation decision configuration parameters.

Once in DETECT state, we use det2active threshold for the number of consecutive hits to transition to ACTIVE state, or det2free threshold of number of consecutive misses to transition back to FREE state.

Once in ACTIVE state, the handling of the MISS (no points associated) is as follow:

- If the target is in the “static zone” AND the target motion model is close to static then the assumption is made that the reason we don’t have detection is because we removed them as “static clutter”. In this case, we increment the miss count, and use static2free threshold to “extend the life expectation” of the static targets.
- If the target is outside the static zone, then the assumption is made that the reason we didn’t get the points is that target is exiting. In this case, we use exit2free threshold to quickly free the exiting targets.
- Otherwise, (meaning target is in the “static zone”, but has non-zero motion in radial projection) we assume that the reason of not having detections is that target got obscured by other targets. In this case, we continue target motion according to the model, and use active2free threshold.

Table 5. State Transitions Parameters

Parameter	Default, TM	Default, PC	Dim	Name
det2activeThre	3U	10U	-	In DETECT state; how many consecutive HIT events needed to transition to ACTIVE state
det2freeThre	3U	5U	-	In DETECT state; how many consecutive MISS events needed to transition to FREE state
active2freeThre	5U	10U	-	In ACTIVE state and NORMAL condition; how many consecutive MISS events needed to transition to FREE state
static2freeThre	5U	100U	-	In ACTIVE state and STATIC condition; how many consecutive MISS events needed to transition to FREE state
exit2freeThre	5U	5U	-	In ACTIVE state and EXIT condition; how many consecutive MISS events needed to transition to FREE state

Gating Parameters

Gating parameters set is used in association process to provide a boundary for the points that can be associated with a given track. These parameters are target-specific.

Table 6. Gating Function Parameters

Parameter	Default, TM	Default, PC	Dim	Description
Volume	16.f	2.f		Gating volume
LengthLimit	12.f	2.f	m	Gating Limit in length
WidthLimit	8.f	2.f	m	Gating Limit in width
VelocityLimit	0.f	0.f	m/s	Gating Limit in velocity (0 – no limit)

Gating volume can be estimated as the volume of the Ellipsoid, computed as

$$V = \frac{4\pi}{3}abc$$

where a , b , and c are the expected target dimensions in range (m), angle (rad), and doppler (m/s).

For example, consider a vehicle as a radar target. For the vehicle center, we could want to reach +/- 4m in range ($a = 8$), +/- 3 degree in azimuth ($b = 6\pi/180$), and +/- 2m/s in radial velocity ($c = 4$), resulting in volume about 16.

In addition to setting the volume of the gating ellipsoid, the limits can be imposed to protect ellipsoid from overstretching. The limits are the function of the geometry and motion of the expected targets. For example, setting WidthLimit to 8m will not allow the gating function to stretch beyond 8m in width.

4.3. Memory Requirements

Initial memory footprint for group tracker implementation with 250 measurement points and 20 tracks is 80KB of data and 20KB of program space.

4.3.1. Data Memory

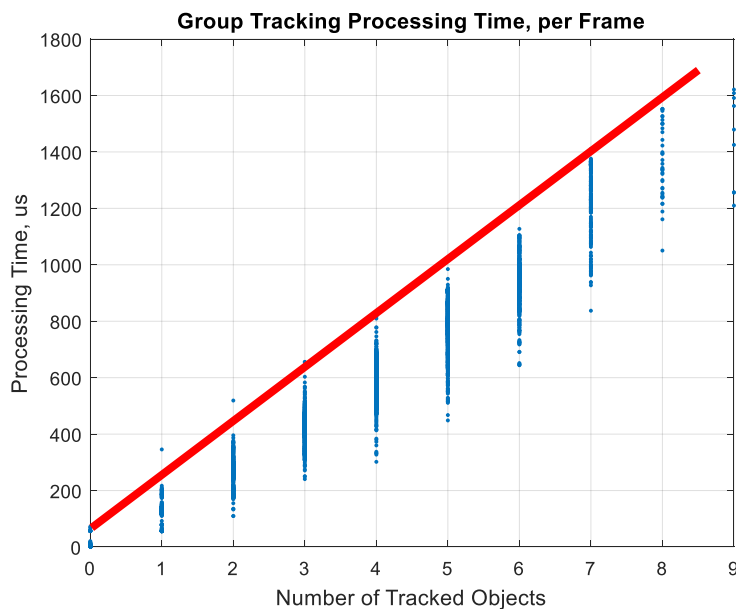
NumTracks	20				
NumPoints	250				
	Size, B	Element Name	Element Type	Size	Num
TrackModule					
	544	inst	GtrackModuleInstance	544	1
	80	inst->hTracks	void *	4	20
	1000	inst->bestScore	float	4	250
	500	inst->bestIndex	uint16_t	2	250
	6	inst->allocIndex	uint16_t	2	3
	240	inst->tidElem	GTrack_ListElem	12	20
	720	inst->targetDesc	GTRACK_targetDesc	36	20
	40	inst->targetInd	uint16_t	2	20
SubTotal	3130				
TrackUnit					
	11520	inst	GtrackUnitInstance	576	20
SubTotal	11520				
Total, Bytes	14650				

4.3.1. Program Memory

	text	data	bss	dec	hex	filename
	212	0	0	212	d4	gtrackListlib.oer4f
	252	0	0	252	fc	gtrackModuleConstants.oer4f
	1980	0	0	1980	7bc	gtrackModuleCreate.oer4f
	1882	0	0	1882	75a	gtrackModuleStep.oer4f
	1182	0	0	1182	49e	gtrackUnitCreate.oer4f
	320	0	0	320	140	gtrackUnitEvent.oer4f
	442	0	0	442	1ba	gtrackUnitPredict.oer4f
	30	0	0	30	1e	gtrackUnitReport.oer4f
	1771	0	0	1771	6eb	gtrackUnitScore.oer4f
	328	0	0	328	148	gtrackUnitStart.oer4f

	196	0	0	196	c4	gtrackUnitStop.oer4f
	2186	0	0	2186	88a	gtrackUnitUpdate.oer4f
	1586	0	0	1586	632	gtrackUtilities.oer4f
	2514	0	0	2514	9d2	matrixMath.oer4f
Total	14881					

4.4. Benchmarks



Tracking processing cost $\approx 200\mu\text{s}$ per tracking object

5. Performance

Tracking performance can be intuitively expressed in three classes of metrics:

1. Tracking Reliability, which shows how many mistakes the tracker made in terms of missing tracks, false positives, mismatches, failures to recover tracks, etc.
2. Tracking Precision, which expresses how well exact features of the correctly tracked objects are estimated,
3. Tracking Resolution, which expresses how much feature separation needed between two objects to be tracked separately

5.1. Tracking Reliability

The test case to measure tracking reliability is simulated randomized longevity test. We modeled tracker performance in traffic monitoring usage case.

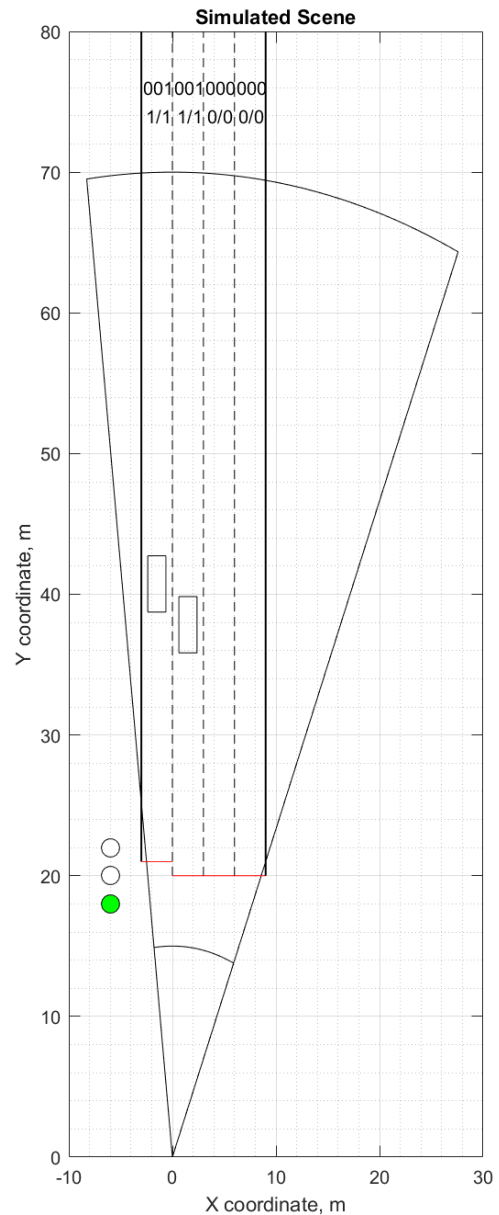
5.1.1. Test Case description

The sensor geometry is illustrated in the picture below. Traffic is simulated by random Poisson process, where vehicles arrive into 4-Lane intersection at random velocities. The intersection is controlled by the traffic light with predetermined latencies for green, yellow, and red signals. The stop line is at 20m from the sensor. Vehicles movement is modeled with following rules:

1. Each vehicle maintains constant velocity unless there is an obstacle in front. The obstacle can be another vehicle in front or either red or yellow traffic light.
2. With obstacle in front, the vehicle will decelerate to match the obstacle speed. The amount of deceleration is bounded to a maximum allowed for the given vehicle type.
3. The vehicles preserve a safety 2m distance.
4. Once obstacle cleared, the vehicles accelerate. The acceleration step and maximum amount of acceleration is configured per vehicle type.

The reliability test simulates at least 10 minutes of traffic.

Sensor output (the point cloud) is randomly generated and is modelled after the test data captured from the moving vehicle. The model includes statistical modeling of number of reflection points, points distribution in range/angle space, and distribution of doppler information in a cloud. Each vehicle at any time instance is represented by a point cloud (the set of reflection points with range/angle and radial velocity information).



Note: Point cloud from multiple vehicles is assumed additive, so we didn't model the point cloud degradation due to multiple objects.

For the point cloud synthesis, we collected reflection points from the field test in two detection layer configurations:

- Configuration A, with “maximum possible” reflections (single range CFAR-CASO, no windowing in doppler dimension, 0.2deg DoA step)
- Configuration B, with about 1/3 of reflections (dual pass range/doppler CFAR-CASO, 0.2deg DoA)

The tracker inputs the points cloud, associates reflection points to the tracked objects, predicts, and models the objects behavior, and outputs estimated object properties to an upper layer.

Each configuration is run in two subcases: with

The upper layer processing estimates the tracking error by comparing the ground truth with tracker output. As a ground truth we consider a centroid of the rectangle of simulated vehicle. For each track, based on global time stamp, we synchronize the ground truth point and tracker output. We compute mean square error individually for all time instances the tracker existed. We declare correct tracking of the object if at any point the mean square error did not exceed the desired threshold, and tracking error event otherwise.

In addition, upper layer processing counts number of objects at 25m lane. This count is compared to a ground truth number as well.

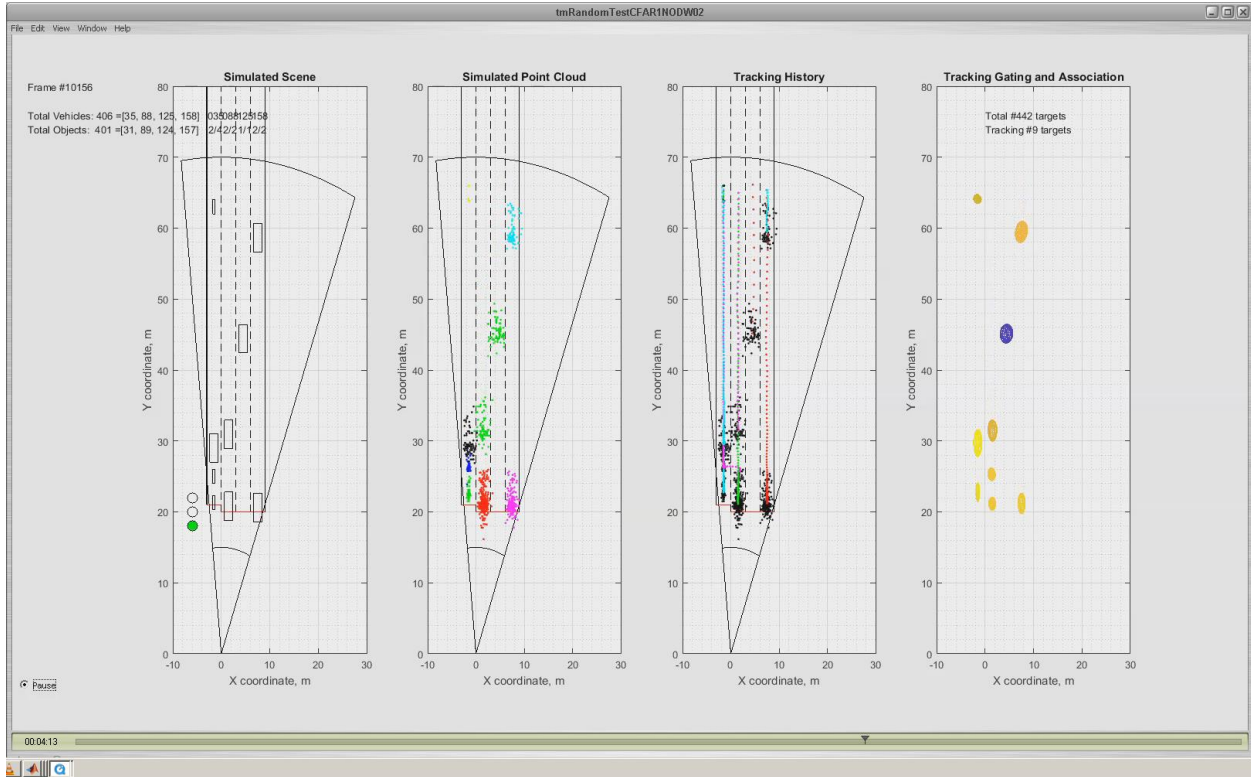
5.1.2. Results

The results are summarized below

Configuration A results

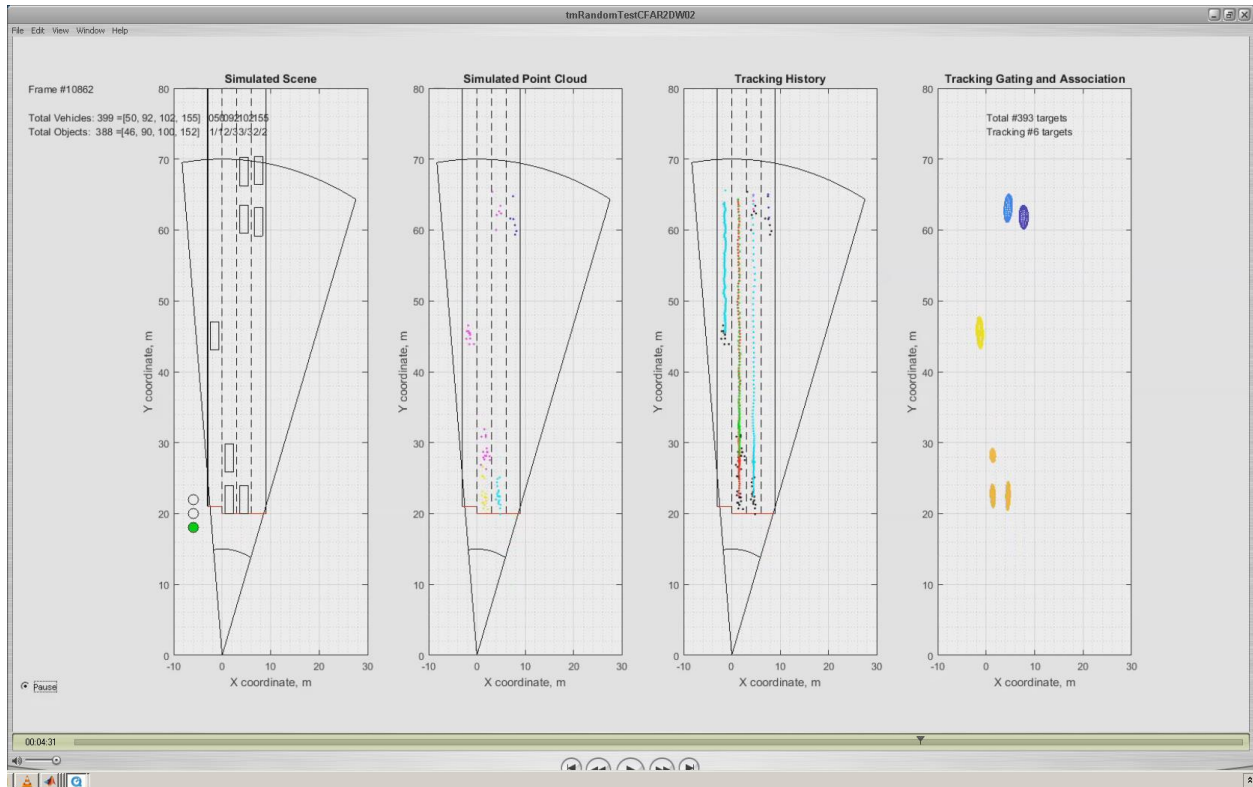
Lane	Number of Vehicles	Number of Counted objects (at 25m)	Counting Reliability, %	Number of correctly tracked objects	Tracking Reliability, %
1	58	58		56	
2	117	122		110	
3	177	184		173	
4	228	233		216	
Total	580	597	99.5	555	95.7

TODO: Due to changes in unrolling functions, the need to Re-run the configuration A tests



Configuration B results

Lane	Number of Vehicles	Number of Counted objects (at 25m)	Counting Reliability	Number of correctly tracked objects	Tracking Reliability, %
1	61	60		57	
2	124	122		110	
3	150	147		126	
4	211	208		194	
Total	546	537	98.4	487	89.4



Test is reproducible with this random seed:

Test outputs (the 10-minute movie) is saved here:

The error analysis shows that most of the errors fall into two cases:

- a) errors due to erroneous split, i.e. at some time instance the tracker observed too much dispersion within a group, and decided to split a group into two distinct objects
- b) errors due to original acquisition, i.e. the track was started with significant deviation from the ground truth, and wasn't able to converge fast enough

5.2. Tracking Precision

Tracking Precision expresses how well exact features of the correctly tracked objects are estimated. This score is independent of tracker ability to identify and follow the objects. For the correctly tracked object, we compute:

1. Position Tracking Precision. This is the measure of expected positional error (mean and deviation) for matched objects
2. Velocity Tracking Precision. This is the measure of expected velocity error (mean and deviation) for matched objects. Shows the ability of the tracker to estimate object velocity
3. Size Tracking Precision. This is the measure of expected dimensional error (mean and deviation) for matched objects. Shows the ability of the tracker to estimate object size

5.2.1. Test Case description

For tracking precision, we use the same randomized longevity test case as described above. We take only the “successfully tracked” objects, and derive the error statistics for those cases, comparing the tracked object with the ground truth.

5.2.2. Results

For successfully tracked objects (95% cases, see the Tracking Reliability Test results), the following was observed:

Configuration A results

Tracking Precision (% of cases)	Error, Standard Deviation at 40m of Range
Latitude (X position), m	
Longitude (Y position), m	
Velocity, Latitudal, m/s	
Velocity, Longitudal, m/s	

Configuration B results

Tracking precision (89.4 % of cases)	Error, Standard Deviation at 40m of Range
Latitude (X position), m	0.11
Longitude (Y position), m	0.36
Velocity, Latitudal, m/s	0.99
Velocity, Longitudal, m/s	0.4

5.3. Tracking Resolution

Tracking Resolution expresses how much of feature separation is needed between two real life objects to be tracked separately. The feature separation can be measured in Range/Angle/Velocity. We designed separate test cases for initial separation, and dynamic (split).

5.3.1. Initial Separation Tests

Test Case description

We designed three separate test cases for Range, Angular, and radial velocity separation. We simulate the vehicle entry with

- a) Range difference (same lane, two vehicles, same velocity, one after each other at given distance),

- b) Angle difference (same range, different lanes, same velocity)
- c) Velocity difference (adjacent lanes, minimal angular difference, different velocities)

For all tests we are seeking a minimal separation needed for a tracker to allocate and successfully track two objects as separate ones in more than 95% of cases.

Results

	Test Case Range	Test Case Angle	Test Case Velocity
Range Difference, m	4	0	0
Angular Separation, deg	0	4	0
Radial Velocity, m/s	0	0	4
Test Success Rate	>95%	>95%	>95%

5.3.2. Dynamic Separation Tests

Test Case description

Separate test is designed, where at entrance, two objects are not distinguishable. The tracker shall allocate a single tracking object. However, due to velocity difference, after few tens of frames, tracker shall be able to trigger a split event and track two targets. We measure the amount of separation required (in Range AND Angle AND Velocity) when tracker performs split in >95% cases.

Results

	Test Case Split
Range Difference, m	4
Angular Separation, deg	4
Radial Velocity, m/s	4
Test Success Rate	>95%

6. References

- [1] <https://github.com/rllabbe/Kalman-and-Bayesian-Filters-in-Python>
- [2] Samuel S. Blackman. Multi-Target Tracking with Radar Applications, Artech House, 1986
- [3] Muhammad Ikram, Nano radar DSP Algorithm Module, Tracking module description

7. Appendix

7.1. Evaluating Partial Derivatives for 2D space tracking

We need to calculate set of partial derivatives to compute Jacobian

$$J_H(s) = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial \dot{x}} & \frac{\partial r}{\partial \dot{y}} & \frac{\partial r}{\partial \ddot{x}} & \frac{\partial r}{\partial \ddot{y}} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial \dot{x}} & \frac{\partial \varphi}{\partial \dot{y}} & \frac{\partial \varphi}{\partial \ddot{x}} & \frac{\partial \varphi}{\partial \ddot{y}} \\ \frac{\partial \dot{r}}{\partial x} & \frac{\partial \dot{r}}{\partial y} & \frac{\partial \dot{r}}{\partial \dot{x}} & \frac{\partial \dot{r}}{\partial \dot{y}} & \frac{\partial \dot{r}}{\partial \ddot{x}} & \frac{\partial \dot{r}}{\partial \ddot{y}} \end{bmatrix}$$

7.1.1. Evaluating range partial derivatives

$$r = \sqrt{x^2 + y^2}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x} (\sqrt{x^2 + y^2}) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y} (\sqrt{x^2 + y^2}) = \frac{y}{\sqrt{x^2 + y^2}}$$

7.1.2. Evaluating azimuth partial derivatives

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x} \left(\tan^{-1} \left(\frac{x}{y} \right) \right)$$

$$\text{Let } w = \tan^{-1} u \text{ with } u = \frac{x}{y}$$

$$\text{Using the chain rule: } \frac{\partial \varphi}{\partial x} = \frac{dw}{du} \frac{\partial u}{\partial x} = \left(\frac{1}{1+u^2} \right) \left(\frac{1}{y} \right) = \frac{1}{1+\left(\frac{x}{y}\right)^2} \frac{1}{y} = \frac{y}{x^2+y^2}$$

$$\text{Using the chain rule: } \frac{\partial \varphi}{\partial y} = \frac{dw}{du} \frac{\partial u}{\partial y} = \left(\frac{1}{1+u^2} \right) \left(\frac{-x}{y^2} \right) = \frac{1}{1+\left(\frac{x}{y}\right)^2} \frac{-x}{y^2} = -\frac{x}{x^2+y^2}$$

7.1.1. Evaluating doppler partial derivatives

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial \dot{r}}{\partial x} = \frac{\partial}{\partial x} \left(\frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} \right) = \dot{x} \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) + y\dot{y} \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{x^2 + y^2}} \right)$$

Using the quotient rule $\frac{\partial}{\partial x} \left(\frac{f}{g} \right) = \frac{\frac{\partial f}{\partial x} g - \frac{\partial g}{\partial x} f}{g^2}$, solving the first part $\frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2+y^2}} \right) = \frac{\sqrt{x^2+y^2} - \frac{x}{\sqrt{x^2+y^2}} x}{(\sqrt{x^2+y^2})^2} = \frac{y^2}{(x^2+y^2)^{3/2}}$.

For the second part, $\frac{\partial}{\partial x} \left(\frac{1}{\sqrt{x^2+y^2}} \right) = -\frac{1}{2} \frac{2x}{(x^2+y^2)^{3/2}} = -\frac{x}{(x^2+y^2)^{3/2}}$.

$$\frac{\partial}{\partial x} \left(\frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} \right) = \dot{x} \frac{y^2}{(x^2 + y^2)^{3/2}} - y\dot{y} \frac{x}{(x^2 + y^2)^{3/2}} = \frac{y(\dot{x}y - \dot{y}x)}{(x^2 + y^2)^{3/2}}$$

Similarly,

$$\begin{aligned} \frac{\partial \dot{r}}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} \right) = \dot{y} \frac{\partial}{\partial y} \left(\frac{y}{\sqrt{x^2 + y^2}} \right) + x\dot{x} \frac{\partial}{\partial y} \left(\frac{1}{\sqrt{x^2 + y^2}} \right) = \dot{y} \frac{x^2}{(x^2 + y^2)^{3/2}} - x\dot{x} \frac{y}{(x^2 + y^2)^{3/2}} \\ &= \frac{x(\dot{y}x - \dot{x}y)}{(x^2 + y^2)^{3/2}} \end{aligned}$$

$$\frac{\partial \dot{r}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left(\frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} \right) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial \dot{r}}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}} \left(\frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} \right) = \frac{y}{\sqrt{x^2 + y^2}}$$

Putting all together:

$$J_H(\mathbf{s}) = \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} & 0 & 0 & 0 & 0 \\ \frac{y}{x^2 + y^2} & -\frac{x}{x^2 + y^2} & 0 & 0 & 0 & 0 \\ \frac{y(\dot{x}y - \dot{y}x)}{(x^2 + y^2)^{3/2}} & \frac{x(\dot{y}x - \dot{x}y)}{(x^2 + y^2)^{3/2}} & \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} & 0 & 0 \end{bmatrix}$$

7.2. Evaluating Partial Derivatives for 3D space tracking

We need to calculate set of partial derivatives to compute Jacobians for 3DV and 3DA models:

$$J_H(s_{3DV}) = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} & \frac{\partial r}{\partial \dot{x}} & \frac{\partial r}{\partial \dot{y}} & \frac{\partial r}{\partial \dot{z}} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} & \frac{\partial \varphi}{\partial \dot{x}} & \frac{\partial \varphi}{\partial \dot{y}} & \frac{\partial \varphi}{\partial \dot{z}} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} & \frac{\partial \theta}{\partial \dot{x}} & \frac{\partial \theta}{\partial \dot{y}} & \frac{\partial \theta}{\partial \dot{z}} \\ \frac{\partial \dot{r}}{\partial x} & \frac{\partial \dot{r}}{\partial y} & \frac{\partial \dot{r}}{\partial z} & \frac{\partial \dot{r}}{\partial \dot{x}} & \frac{\partial \dot{r}}{\partial \dot{y}} & \frac{\partial \dot{r}}{\partial \dot{z}} \end{bmatrix}$$

and

$$J_H(s_{3DA}) = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} & \frac{\partial r}{\partial \dot{x}} & \frac{\partial r}{\partial \dot{y}} & \frac{\partial r}{\partial \dot{z}} & \frac{\partial r}{\partial \ddot{x}} & \frac{\partial r}{\partial \ddot{y}} & \frac{\partial r}{\partial \ddot{z}} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} & \frac{\partial \varphi}{\partial \dot{x}} & \frac{\partial \varphi}{\partial \dot{y}} & \frac{\partial \varphi}{\partial \dot{z}} & \frac{\partial \varphi}{\partial \ddot{x}} & \frac{\partial \varphi}{\partial \ddot{y}} & \frac{\partial \varphi}{\partial \ddot{z}} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} & \frac{\partial \theta}{\partial \dot{x}} & \frac{\partial \theta}{\partial \dot{y}} & \frac{\partial \theta}{\partial \dot{z}} & \frac{\partial \theta}{\partial \ddot{x}} & \frac{\partial \theta}{\partial \ddot{y}} & \frac{\partial \theta}{\partial \ddot{z}} \\ \frac{\partial \dot{r}}{\partial x} & \frac{\partial \dot{r}}{\partial y} & \frac{\partial \dot{r}}{\partial z} & \frac{\partial \dot{r}}{\partial \dot{x}} & \frac{\partial \dot{r}}{\partial \dot{y}} & \frac{\partial \dot{r}}{\partial \dot{z}} & \frac{\partial \dot{r}}{\partial \ddot{x}} & \frac{\partial \dot{r}}{\partial \ddot{y}} & \frac{\partial \dot{r}}{\partial \ddot{z}} \end{bmatrix}$$

7.2.1. Evaluating range partial derivatives

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x} \left(\sqrt{x^2 + y^2 + z^2} \right) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y} \left(\sqrt{x^2 + y^2 + z^2} \right) = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z} \left(\sqrt{x^2 + y^2 + z^2} \right) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

7.2.2. Evaluating azimuth partial derivatives

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x} \left(\tan^{-1} \left(\frac{x}{y} \right) \right)$$

Let $w = \tan^{-1} u$ with $u = \frac{x}{y}$

Using the chain rule: $\frac{\partial \phi}{\partial x} = \frac{dw}{du} \frac{\partial u}{\partial x} = \left(\frac{1}{1+u^2}\right) \left(\frac{1}{y}\right) = \frac{1}{1+\left(\frac{x}{y}\right)^2} \frac{1}{y} = \frac{y}{x^2+y^2}$

Using the chain rule: $\frac{\partial \phi}{\partial y} = \frac{dw}{du} \frac{\partial u}{\partial y} = \left(\frac{1}{1+u^2}\right) \left(\frac{-x}{y^2}\right) = \frac{1}{1+\left(\frac{x}{y}\right)^2} \frac{-x}{y^2} = -\frac{x}{x^2+y^2}$

$$\frac{\partial \phi}{\partial z} = 0$$

7.2.1. Evaluating elevation partial derivatives

$$\theta = \tan^{-1} \left(\frac{z}{\sqrt{x^2 + y^2}} \right)$$

Let $w = \tan^{-1} u$ with $u = \frac{z}{\sqrt{x^2+y^2}}$

Using the chain rule: $\frac{\partial \theta}{\partial x} = \frac{dw}{du} \frac{\partial u}{\partial x} = \left(\frac{1}{1+u^2}\right) \frac{\partial u}{\partial x} = \frac{1}{1+\frac{z^2}{x^2+y^2}} \left(-\frac{1}{2}\right) 2x \frac{z}{(x^2+y^2)^{3/2}} = -\frac{x^2+y^2}{x^2+y^2+z^2} \frac{xz}{(x^2+y^2)^{3/2}} =$
 $-\frac{x}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} = -\frac{x}{r^2} \tan \theta$

Using the chain rule: $\frac{\partial \theta}{\partial y} = \frac{dw}{du} \frac{\partial u}{\partial y} = \left(\frac{1}{1+u^2}\right) \frac{\partial u}{\partial y} = \frac{1}{1+\frac{z^2}{x^2+y^2}} \left(-\frac{1}{2}\right) 2y \frac{z}{(x^2+y^2)^{3/2}} = -\frac{x^2+y^2}{x^2+y^2+z^2} \frac{yz}{(x^2+y^2)^{3/2}} =$
 $-\frac{y}{(x^2+y^2+z^2)\sqrt{x^2+y^2}} = -\frac{y}{r^2} \tan \theta$

Using the chain rule: $\frac{\partial \theta}{\partial z} = \frac{dw}{du} \frac{\partial u}{\partial z} = \left(\frac{1}{1+u^2}\right) \frac{\partial u}{\partial z} = \frac{1}{1+\frac{z^2}{x^2+y^2}} \frac{1}{\sqrt{x^2+y^2}} = \frac{x^2+y^2}{x^2+y^2+z^2} \frac{1}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2}}{(x^2+y^2+z^2)} =$
 $\frac{\sqrt{x^2+y^2}}{r^2}$

7.2.2. Evaluating Doppler partial derivatives

$$\dot{r} = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial \dot{r}}{\partial x} = \frac{\partial}{\partial x} \left(\frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2 + y^2 + z^2}} \right) = \dot{x} \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) + (y\dot{y} + z\dot{z}) \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right);$$

Using the quotient rule $\frac{\partial}{\partial x} \left(\frac{f}{g} \right) = \frac{\frac{\partial f}{\partial x} g - \frac{\partial g}{\partial x} f}{g^2}$, solving $\frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2+y^2+z^2}} \right) = \frac{\sqrt{x^2+y^2+z^2} - \frac{x}{\sqrt{x^2+y^2+z^2}} x}{x^2+y^2+z^2} =$
 $\frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}};$

Using chain rule, $\frac{\partial}{\partial x} \left(\frac{1}{\sqrt{x^2+y^2+z^2}} \right) = -\frac{1}{2} \frac{2x}{(x^2+y^2+z^2)^{3/2}} = -\frac{x}{(x^2+y^2+z^2)^{3/2}};$

$$\begin{aligned}\frac{\partial}{\partial x} \left(\frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2 + y^2 + z^2}} \right) &= \dot{x} \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}} - (y\dot{y} + z\dot{z}) \frac{x}{(x^2 + y^2 + z^2)^{3/2}} = \\ &= \frac{y(\dot{x}y - \dot{y}x) + z(\dot{x}z - \dot{z}x)}{(x^2 + y^2 + z^2)^{3/2}};\end{aligned}$$

Due to symmetry:

$$\frac{\partial \dot{r}}{\partial y} = \frac{x(\dot{y}x - \dot{x}y) + z(\dot{y}z - \dot{z}y)}{(x^2 + y^2 + z^2)^{3/2}};$$

$$\frac{\partial \dot{r}}{\partial z} = \frac{x(\dot{z}x - \dot{x}z) + y(\dot{z}y - \dot{y}z)}{(x^2 + y^2 + z^2)^{3/2}};$$

$$\frac{\partial \dot{r}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left(\frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{x}{\sqrt{x^2 + y^2 + z^2}};$$

$$\frac{\partial \dot{r}}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}} \left(\frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{y}{\sqrt{x^2 + y^2 + z^2}};$$

$$\frac{\partial \dot{r}}{\partial \dot{z}} = \frac{\partial}{\partial \dot{z}} \left(\frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{z}{\sqrt{x^2 + y^2 + z^2}};$$

Putting all together:

$$\begin{aligned}J_H(S_{3DV}) &= \\ &= \begin{bmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} & 0 & 0 & 0 \\ \frac{y}{x^2 + y^2} & -\frac{x}{x^2 + y^2} & 0 & 0 & 0 & 0 \\ -\frac{x}{r^2} \frac{z}{\sqrt{x^2 + y^2}} & -\frac{y}{r^2} \frac{z}{\sqrt{x^2 + y^2}} & \frac{\sqrt{x^2 + y^2}}{r^2} & 0 & 0 & 0 \\ \frac{y(\dot{x}y - \dot{y}x) + z(\dot{x}z - \dot{z}x)}{r^3} & \frac{x(\dot{y}x - \dot{x}y) + z(\dot{y}z - \dot{z}y)}{r^3} & \frac{x(\dot{z}x - \dot{x}z) + y(\dot{z}y - \dot{y}z)}{r^3} & \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{bmatrix};\end{aligned}$$

$$\begin{aligned}J_H(S_{3DA}) &= \\ &= \begin{bmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{y}{x^2 + y^2} & -\frac{x}{x^2 + y^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{x}{r^2} \frac{z}{\sqrt{x^2 + y^2}} & -\frac{y}{r^2} \frac{z}{\sqrt{x^2 + y^2}} & \frac{\sqrt{x^2 + y^2}}{r^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{y(\dot{x}y - \dot{y}x) + z(\dot{x}z - \dot{z}x)}{r^3} & \frac{x(\dot{y}x - \dot{x}y) + z(\dot{y}z - \dot{z}y)}{r^3} & \frac{x(\dot{z}x - \dot{x}z) + y(\dot{z}y - \dot{y}z)}{r^3} & \frac{x}{r} & \frac{y}{r} & \frac{z}{r} & 0 & 0 & 0 \end{bmatrix};\end{aligned}$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

7.3. Orthogonal projection of an Ellipsoid E onto a given line

We consider a given line \mathcal{L} , parameterized by a scalar s , defined by

$$\mathcal{L} \stackrel{\text{def}}{=} \{x | x = x_0 + s\mathbf{v}\},$$

where x_0 is a given point and \mathbf{v} is a given non-zero vector, see Figure 10.

Given any point x in the space, based on orthogonality of $(x - x_0) - s\mathbf{v}$ and \mathbf{v} , its orthogonal projection s onto \mathcal{L} corresponds to

$$s = \frac{\mathbf{v}^T(x - x_0)}{\mathbf{v}^T\mathbf{v}}$$

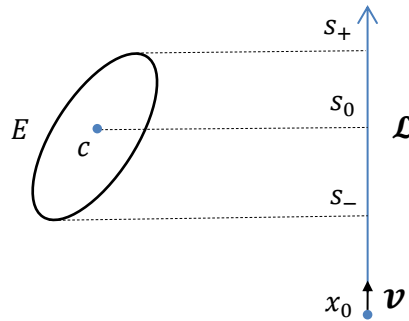


Figure 10. The orthogonal projection of the ellipsoid E onto the line \mathcal{L} is the interval $[S_-, S_+]$,

Now, the ellipsoid E is given by

$$E = \{x | x = c + L^{-T}y, \quad \|y\| < 1\}$$

Thus the projection of points in E correspond to values of s

$$s = \frac{\mathbf{v}^T(c + L^{-T}y - x_0)}{\mathbf{v}^T\mathbf{v}} = s_0 + w^T y, \quad \|y\| < 1$$

where

$$s_0 = \frac{\mathbf{v}^T(c - x_0)}{\mathbf{v}^T\mathbf{v}}$$

$$w = \frac{L^{-1}v}{v^T v}$$

Given the condition $\|y\| < 1$, it is evident that the orthogonal projection of E onto \mathcal{L} corresponds to the interval $[s_-, s_+]$ with

$$s_{\pm} = s_0 \pm |w|$$

And the projection length is

$$s_{\pm} = s_+ - s_- = 2|w|$$

The projections onto orthogonal axes

$$S_{x,y,z} = 2|L^{-1}v_{x,y,z}|$$