Derivation of a FMCW Beat Frequency

Given a transmitted signal with a linear modulated frequency

$$V_t = a_1 \cos(\phi(t)) \tag{1}$$

where a_1 is an amplitude constant and $\phi(t)$ is a phase function of time, (t) being defined as

$$\phi(t) = \omega_o t + \frac{1}{2} m t^2 + \phi_o \tag{2}$$

where $m = 2\pi \frac{BW}{T_p}$ with BW is the bandwidth of the signal, T_p is the time period of the frequency ramp and ϕ_o is an arbitrary phase constant. The initial radian frequency is $\omega_o = 2\pi f_o$ with f_o being the initial frequency of the ramp. The radian frequency of this signal is then

$$\frac{d\phi(t)}{dt} = \omega = \omega_o + mt \tag{3}$$

The received signal is then

$$V_r = a_2 \cos(\phi(t - T_d)) \tag{4}$$

where a_1 is an amplitude constant and $\phi(t - T_d)$ is the transmitted phase function delayed by $T_d = 2R/c$ which is due the the propagation of the signal through a one way distance of R at the speed of light, c.

The IF signal at a mixer is then

$$V_b = V_t V_r = a_1 a_2 a_3 \cos\left[\omega_o t + \frac{1}{2} m t^2 + \phi_o\right] \cos\left[\omega_o t + \frac{1}{2} m t^2 + \phi_o - T_d + \phi_1\right] \tag{5}$$

where a_3 and ϕ_1 are amplitude and phase constants associated with the mixer.

Reducing the product of the cosines yields

$$V_b = \frac{1}{2} a_1 a_2 a_3 \cos\left[mT_d t + \omega_o T_d + \frac{m}{2} T_d^2 - \phi_1 + \text{higher order terms}\right] \tag{6}$$

or

$$V_b = \frac{1}{2} a_1 a_2 a_3 \cos\left[\omega_b t + \phi_c - \phi_1\right] \tag{7}$$

where $\omega_b = 2\pi f_b$ with f_b is the beat frequency is given by

$$\frac{BW}{T_p} \frac{2R}{c} \tag{8}$$

and phase constant is given by

$$\phi_c = 2\pi \frac{2R}{\lambda_o} \tag{9}$$

with $\lambda_o = c/f_o$. The term $\frac{m}{2} T_d^2$ is very small and dropped.